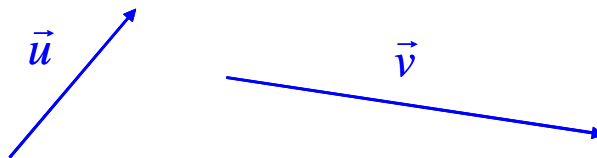


CHAPTER 4

VECTORS

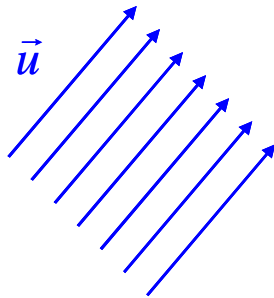
Before we go any further, we must talk about vectors. They are such a useful tool for the things to come. The concept of a vector is deeply rooted in the understanding of physical mechanics and how physicists view forces in the universe. A force tends to act in a particular direction and with its own particular magnitude, and this is the basis for the notion of a vector. A vector is, thus, generally thought of as a quantity that possesses both direction and magnitude. In terms of notation, we often put a small arrow above a letter to indicate that we want it to represent a vector. Hence, we write  $\vec{u}$  or  $\vec{v}$  to indicate that we are speaking of vectors. Graphically, we draw arrows to represent vectors. In this geometric definition, the length of the arrow corresponds to the magnitude of the vector, and the direction the arrow points in represents the direction of the vector. Thus, we'll also work with graphical representations such as the ones below.



An interesting consequence of our definition of a vector as something possessing magnitude and direction is that vectors aren't defined by their location in space. In other words, two arrows that point in the same direction and have the same magnitude

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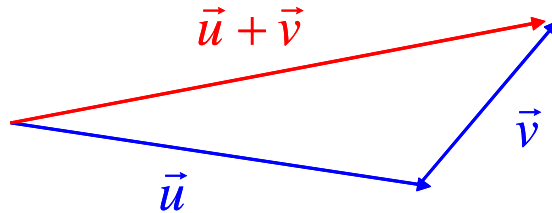
represent the same vector regardless of their location. This may seem strange not to tie a vector to a location, but it actually works out better this way in practice because to add vectors together geometrically, we are going to have to move them around. Think of it like this. Suppose vectors are Lego pieces and that we add two pieces by connecting them together. In that case, we wouldn't consider a Lego piece as losing its identity simply because we picked it up and moved it around. It's the same way with vectors. If, in a geometric sense, I pick one up and move it to a different location, then it's still the same vector as long as it has the same length and points in the same direction. Hence, each of the arrows below represents the same vector  $\vec{u}$ .



Again, though, you might object by saying that when something pushes against you, that force is not only acting in a particular direction with a particular magnitude, it's also being applied at a particular location in space. Well, that's absolutely true, and we have another concept to cover that case. When we do tie vectors to locations in space, we call the resulting configuration a vector field, and we'll deal extensively with vector fields much later on. But for now, we'll just think of vectors like the Lego pieces that we can move around and place wherever we want.

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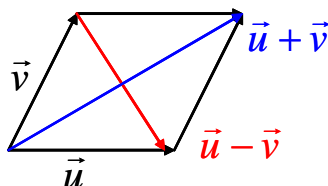
To add two vectors together geometrically, we simply place the starting point of one vector at the stopping point of the other vector, and then draw a single vector from the first starting point to the ultimate stopping point. The end result looks like this.



Also, given a vector  $\vec{u}$ , we define  $-\vec{u}$  as the vector with the same magnitude, but pointing in the opposite direction. Additionally, we define something such as  $2\vec{u}$  as the vector pointing in the same direction as  $\vec{u}$ , but with twice the length. When we multiply a vector times a number, we call that number a scalar, and in general, if one vector is a scalar multiple of another, then we say that the two vectors are parallel.

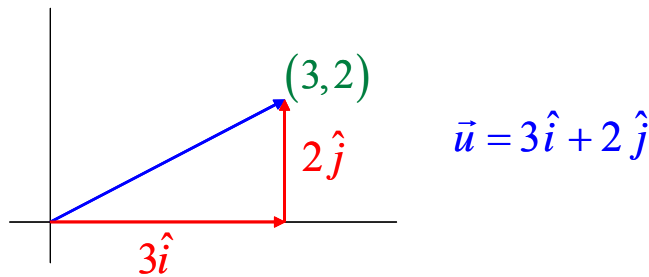


One final diagram worth looking at is the parallelogram formed by two vectors. In this drawing, one of the diagonals represents the sum of the two vectors, and the other diagonal represents the difference.



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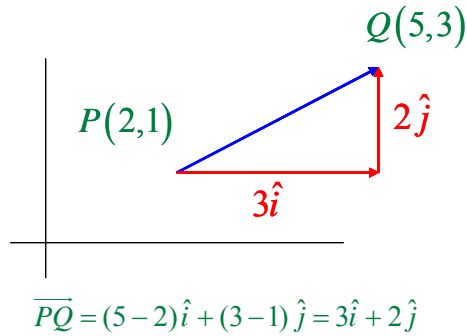
Well, drawing little diagrams made of arrows is loads of fun, but a very tedious way to do math. We really need a more algebraic way to denote and do operations with vectors, and to accomplish this task we'll define three very special unit vectors, i.e. vectors of length one unit. We'll call these vectors  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$ . The vector  $\hat{i}$  has length 1 and points in the direction of the positive  $x$ -axis, the vector  $\hat{j}$  has length 1 and points in the direction of the positive  $y$ -axis, and the vector  $\hat{k}$  has length 1 and points in the direction of the positive  $z$ -axis. The diagram below now illustrates how, in 2-dimensions, we might represent a vector as the sum of its  $\hat{i}$  and  $\hat{j}$  components.



Notice, too, that if we place the starting point of a vector at the origin, then the coordinates of the stopping point correspond to the  $\hat{i}$  and  $\hat{j}$  components of the vector. Also, the more formal terms for starting and stopping points of a vector are either initial point and terminal point or tail and head of the vector. When we place a vector so that its initial point is at the origin, then we call that vector a position vector, and there is a natural association between the terminal point of a position vector and the vector's  $\hat{i}$  and  $\hat{j}$  components. If instead of starting our vector at the origin, we begin at a point  $P$  and terminate at a point  $Q$ , then we call that vector the

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displacement vector  $\overline{PQ}$ , and to get the  $\hat{i}$  and  $\hat{j}$  components of that vector we just do some subtraction with the corresponding  $x$  and  $y$  coordinates.



Everything we've done above easily extends into three dimensions by merely adding in the  $\hat{k}$  component for a true 3-dimensional representation. Furthermore, we can now easily do arithmetic with vectors just by performing the operations on their components. In other words, to add or subtract two vectors, just add or subtract their  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  components, and to multiply a vector times a scalar, just multiply each component by that scalar. Below are a few simple examples.

Examples: Let  $\vec{u} = 2\hat{i} + 3\hat{j} - 5\hat{k}$  and let  $\vec{v} = -\hat{i} + 4\hat{j} + 2\hat{k}$ . Then,

$$\vec{u} + \vec{v} = (2-1)\hat{i} + (3+4)\hat{j} + (-5+2)\hat{k} = \hat{i} + 7\hat{j} - 3\hat{k}$$

$$\vec{u} - \vec{v} = (2+1)\hat{i} + (3-4)\hat{j} + (-5-2)\hat{k} = 3\hat{i} - \hat{j} - 7\hat{k}$$

$$2\vec{u} = 2(2\hat{i} + 3\hat{j} - 5\hat{k}) = 4\hat{i} + 6\hat{j} - 10\hat{k}$$

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Let's now talk about how we might find the length of a vector algebraically. Again, let  $\vec{u} = 2\hat{i} + 3\hat{j} - 5\hat{k}$ . If we think of this as a position vector that has its initial point at the origin, then its terminal point is  $(2, 3, -5)$ , and thus, its length is just the distance that this point is from the origin. By the distance formula (which follows from the Pythagorean Theorem) we get that this is  $\sqrt{2^2 + 3^2 + (-5)^2} = \sqrt{4 + 9 + 25} = \sqrt{38}$ . We usually denote the length of a vector by putting either absolute value signs or double absolute value signs around the vector to form either  $|\vec{u}|$  or  $\|\vec{u}\|$ . I'm going to adopt the latter notation because I think it looks so much cooler! Thus, if  $\vec{u} = 2\hat{i} + 3\hat{j} - 5\hat{k}$ , then  $\|\vec{u}\| = \sqrt{38}$ . Additionally, there are going to be many times that we might want to construct a unit vector pointing in the same direction as our original vector. To accomplish this, all we need to do is divide our original vector by its length, and the result will be a vector of length 1 pointing in the same direction.

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\|\vec{u}\|} \cdot \vec{u} = \frac{1}{\sqrt{38}} (2\hat{i} + 3\hat{j} - 5\hat{k}) = \frac{2}{\sqrt{38}}\hat{i} + \frac{3}{\sqrt{38}}\hat{j} - \frac{5}{\sqrt{38}}\hat{k}$$

We also like to call this unit vector *the direction of the vector*  $\vec{u}$ . Generally, the easiest way to talk about the direction of any particular vector is to specify the unit vector that points the same way.

One more thing I should probably mention before moving on is an alternate notation that one can use for vectors written in terms of their  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  components.

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Instead of writing it as above, we can merely enclose the components inside brackets to denote the vector as in  $\left\langle \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, -\frac{5}{\sqrt{38}} \right\rangle = \frac{2}{\sqrt{38}}\hat{i} + \frac{3}{\sqrt{38}}\hat{j} - \frac{5}{\sqrt{38}}\hat{k}$ . I don't use this notation all that often, but it does sometimes come in handy. Additionally, some people like to enclose the components inside parentheses rather than brackets, but this can cause some minor confusion regarding whether we are talking about a vector or a point.

Well, we're on a roll now! Representing a vector in terms of components makes vector arithmetic a lot easier, and we don't need to try and draw arrows in three dimension in order to do it. Now let's look at something more advanced. Namely, how we might multiply vectors together.

This may be hard to believe, but every now and then you will come across something in mathematics that looks totally weird, and you'll wonder how people ever came up with something like that. Vector multiplication is going to be a good example of such weirdness. We actually have two ways of multiplying vectors, and they are known, respectively, as the dot product and the cross product. Both of these methods were chanced upon in 1773 by the great mathematician Joseph Lagrange. He was working on a problem involving tetrahedrons, and apparently needed to utilize what we call today the dot product and the cross product in order to arrive at his solution. For

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Lagrange these two product were most likely simply a means to an end in the course of his problem solving, but as often happens in mathematics, one person discovers something, and then others find a plethora of applications to other situations.

Consequently, mathematicians occasionally chance upon something that looks weird and unusual, but if we find a lot of uses for it, then we keep it! And that's certainly the case with the dot product and the cross product.

So let's begin with the dot product (denoted by  $\vec{a} \cdot \vec{b}$  and read " $\vec{a}$  dot  $\vec{b}$ "). What exactly is the dot product? Well, we'll define it this way.

Definition: If  $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$  and  $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$ , then the dot product is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

What the #\$@\*^? Well, trust me. This is going to turn out to be very useful. But first, a couple of comments. Notice that the dot product of two vectors is going to be a scalar, a number, and not another vector. Thus, if  $\vec{u} = 2\hat{i} + 3\hat{j} - 5\hat{k}$  and  $\vec{v} = -\hat{i} + 4\hat{j} + 2\hat{k}$ , then  $\vec{u} \cdot \vec{v} = (2)(-1) + (3)(4) + (-5)(2) = -2 + 12 - 10 = 0$ . I wasn't really expecting the dot product to equal zero in this example, but we'll find out soon why having a dot product equal to zero is especially cool. Another thing we should notice about the dot product is that it is commutative. In other words,  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ . We can



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prove this just by appealing to the definition and using the fact that multiplication of ordinary numbers is commutative. Thus,

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = b_1 a_1 + b_2 a_2 + b_3 a_3 = \vec{b} \cdot \vec{a}. \text{ Another interesting fact is that we also}$$

think of the dot product as an example of matrix multiplication. In other words, if

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \text{ and } \vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}, \text{ then}$$

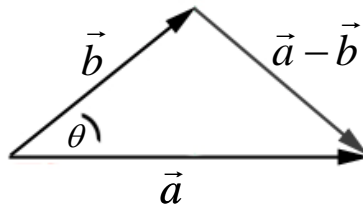
$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Now we need to prove a theorem that will reveal some of the usefulness of the dot product.

**Theorem:** If  $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$  and  $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$ , then  $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$ , where

$\theta$  is the angle between the two vectors.

**Proof:** Let's start by looking at the triangle formed by the vectors  $\vec{a}$  and  $\vec{b}$ . It might look something like the following.



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Also, according to the Law of Cosines, we'll have  $\|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$ .

But this implies that

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta.$$

Expanding the left side of this equation yields

$$a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2 + a_3^2 - 2a_3b_3 + b_3^2$$

$$= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta. \text{ Next, subtracting like terms from each}$$

side gives us  $-2a_1b_1 - 2a_2b_2 - 2a_3b_3 = -2\|\vec{a}\|\|\vec{b}\|\cos\theta$ . And finally, dividing by -2 results

in  $a_1b_1 + a_2b_2 + a_3b_3 = \|\vec{a}\|\|\vec{b}\|\cos\theta$ . Therefore,  $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$ . ■

The immediate implication of this theorem is that we can use the dot product to find

the angle between two vectors. In particular,  $\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right)$ . For

example, if we go back to our previous vectors  $\vec{u} = 2\hat{i} + 3\hat{j} - 5\hat{k}$  and  $\vec{v} = -\hat{i} + 4\hat{j} + 2\hat{k}$ ,

then the angle between them is

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) = \cos^{-1}\left(\frac{0}{\sqrt{38}\sqrt{21}}\right) = \cos^{-1}(0) = 90^\circ = \frac{\pi}{2}. \text{ In other words, the two}$$

vectors are actually perpendicular to one another, and this leads to an important corollary.

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Corollary: Two vectors  $\vec{a}$  and  $\vec{b}$  are perpendicular if and only if  $\vec{a} \cdot \vec{b} = 0$ .

We now have an application of the dot product that is far from trivial. We can now easily find the angle between two vectors, and we can easily determine when two vectors are perpendicular (or orthogonal, as we also say). Another quick application of the dot product is that  $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$ . In other words, the dot product of a vector with itself is equal to the square of its length. The proof is easy and is left to you. Just write everything in component form and do the math!

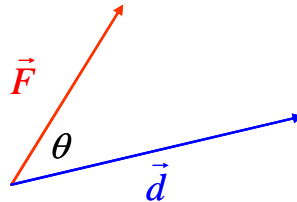
If the angle between two vectors is either 0 or  $\pi$  (equivalently,  $0^\circ$  or  $180^\circ$ ), then we say that the vectors are parallel. Another way to state this is to say that two vectors are parallel if and only if one vector is a scalar multiple of the other. For example,  $\vec{u} = \hat{i} + \hat{j} + \hat{k}$ ,  $\vec{v} = 2\vec{u} = 2\hat{i} + 2\hat{j} + 2\hat{k}$ , and  $\vec{w} = -\vec{u} = -\hat{i} - \hat{j} - \hat{k}$  are all parallel to one another.

Another application of the dot product that we'll use extensively later on is as a tool to compute work. If you've taken some physics, then you know that physicists have their own definition of work. The definition they generally use is  $\text{work} = \text{force} \times \text{distance}$ . This usually, but not always, corresponds well to our common sense notion of work. For example, if you have to use a lot of force to push

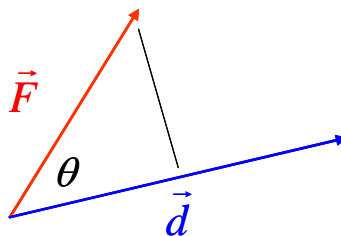
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something a great distance, that's a lot of work! But suppose for a moment that the direction in which the force is applied is not the same as the direction in which you are moving the object. What do you do then? That's where vectors help us!

Suppose we have two vectors  $\vec{F} = \hat{i} + 2\hat{j}$  and  $\vec{d} = 4\hat{i} + \hat{j}$ .



The vector  $\vec{d} = 4\hat{i} + \hat{j}$ , in this case, represents the path that a force  $\vec{F} = \hat{i} + 2\hat{j}$  is going to move something along, and we want to compute the work done. If you wish, think of the force as being measured in pounds and the distance as being measured in feet. The work then, as a product of the two, will have units of foot-pounds. What we need to do now is to figure out the component of our force vector that is parallel to the vector  $\vec{d} = 4\hat{i} + \hat{j}$ .



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Using some trigonometry, that component is

$$\text{comp}_{\vec{d}}\vec{F} = \|\vec{F}\|\cos\theta = \frac{\|\vec{F}\|\|\vec{d}\|\cos\theta}{\|\vec{d}\|} = \frac{\vec{F}\cdot\vec{d}}{\|\vec{d}\|} = \frac{6}{\sqrt{17}} = \frac{6\sqrt{17}}{17}. \text{ The work done is now simply}$$

the component of  $\vec{F}$  along the vector  $\vec{d}$  times the length of the vector  $\vec{d}$ . In other

words,  $\text{work} = (\text{comp}_{\vec{d}}\vec{F})\|\vec{d}\| = \left(\frac{\vec{F}\cdot\vec{d}}{\|\vec{d}\|}\right)\|\vec{d}\| = \vec{F}\cdot\vec{d} = 6 \text{ foot-pounds}$ . The bottom line is that

if a constant force  $\vec{F}$  is moving an object in a straight line represented by a vector  $\vec{d}$ ,

then the work done is  $\vec{F}\cdot\vec{d}$ . Remember this one! It's important. Also, if we take the

component of  $\vec{F}$  along  $\vec{d}$  and multiply it by a unit vector in the direction of  $\vec{d}$ , then

we call that the projection of  $\vec{F}$  onto  $\vec{d}$ . In particular, for the vectors  $\vec{F} = \hat{i} + 2\hat{j}$  and

$\vec{d} = 4\hat{i} + \hat{j}$  we have

$$\text{proj}_{\vec{d}}\vec{F} = (\|\vec{F}\|\cos\theta)\frac{\vec{d}}{\|\vec{d}\|} = \frac{\vec{F}\cdot\vec{d}}{\|\vec{d}\|}\cdot\frac{\vec{d}}{\|\vec{d}\|} = \left(\frac{\vec{F}\cdot\vec{d}}{\|\vec{d}\|^2}\right)\vec{d} = \left(\frac{\vec{F}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\right)\vec{d} = \frac{24}{17}\hat{i} + \frac{6}{17}\hat{j}.$$

Now let's look at how we define the cross product,  $\vec{a}\times\vec{b}$ , and trust me, this is really

going to look crazy! Also, we read  $\vec{a}\times\vec{b}$  as " $\vec{a}$  cross  $\vec{b}$ ."

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**Definition:** If  $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$  and  $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$ , then

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k} \\ &= (a_2 b_3 - a_3 b_2) \hat{i} - (a_1 b_3 - a_3 b_1) \hat{j} + (a_1 b_2 - a_2 b_1) \hat{k}.\end{aligned}$$

Cool! Three things we should notice right now,

1. The cross product is defined as the determinant of a matrix.
2. The cross product of two vectors results in another vector.
3. We have no idea at this point why this is going to be useful.

Let's start alleviating the problem highlighted in the last item above by proving a few theorems.

**Theorem:** The cross product of two vectors  $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$  and  $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$  is perpendicular to both vectors.

**Proof:** Be definition,  $\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2) \hat{i} - (a_1 b_3 - a_3 b_1) \hat{j} + (a_1 b_2 - a_2 b_1) \hat{k}$ . To show that  $\vec{a}$  is perpendicular to this vector, we just need to show that their dot product is equal

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to zero. Hence,

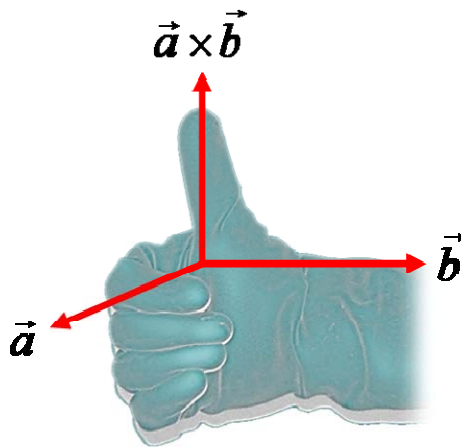
$$\vec{a} \cdot (\vec{a} \times \vec{b}) = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot ((a_2 b_3 - a_3 b_2) \hat{i} - (a_1 b_3 - a_3 b_1) \hat{j} + (a_1 b_2 - a_2 b_1) \hat{k})$$

$= a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - a_3 a_2 b_1 = 0$ . A similar calculation shows that

$\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$ , and so it follows that the cross product  $\vec{a} \times \vec{b}$  is perpendicular to both  $\vec{a}$

and  $\vec{b}$ . ■

If our vectors  $\vec{a}$  and  $\vec{b}$  are not parallel, then they are going to define a plane, and the cross product  $\vec{a} \times \vec{b}$  will be perpendicular to this plane. However, we still want more information because, for example, a vector perpendicular to a horizontal plane could point either up or down. Can we possibly determine in an easy way which direction our cross product will point in? Fortunately, the answer is yes. We just use what we call the right-hand rule.



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If we point the fingers of our right hand in the direction of vector  $\vec{a}$  and curl our fingers towards vector  $\vec{b}$ , then our thumb will point in the direction of  $\vec{a} \times \vec{b}$ . Notice, in particular, that if we apply this right-hand rule to the cross product  $\vec{b} \times \vec{a}$ , then the resulting vector points in the opposite direction from  $\vec{a} \times \vec{b}$ . This is very important because it tells us that, unlike the dot product, the cross product is not commutative. In other words,  $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ . Thus, the order in which you do these things makes a difference. Also, the correct relationship between these cross products is

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a}).$$

Now let's look at some more theorems.

Theorem:  $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin \theta$ , where  $\theta$  is the angle between the two vectors.

Proof: The proof is amazingly simple!



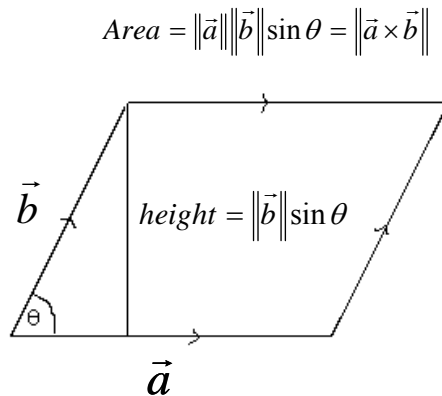
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$$\begin{aligned}
 \|\vec{a} \times \vec{b}\|^2 &= (a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2 \\
 &= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_1^2b_3^2 - 2a_1a_3b_1b_3 + a_3^2b_1^2 \\
 &\quad + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\
 &= a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 \\
 &\quad - 2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3 \\
 &= a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 \\
 &\quad - 2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3 - a_1^2b_1^2 - a_2^2b_2^2 - a_3^2b_3^2 \\
 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
 &= \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 - (\|\vec{a}\| \|\vec{b}\| \cos \theta)^2 \\
 &= \|\vec{a}\|^2 \|\vec{b}\|^2 - \|\vec{a}\|^2 \|\vec{b}\|^2 \cos^2 \theta = \|\vec{a}\|^2 \|\vec{b}\|^2 (1 - \cos^2 \theta) = \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2 \theta.
 \end{aligned}$$

Thus,  $\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2 \theta$ , and since  $\sin \theta \geq 0$  for  $0 \leq \theta \leq \pi$ , we have that

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta. \quad \text{Any questions???} \blacksquare$$

One application of this result is a formula in terms of vectors for the area of a parallelogram. The picture below should be pretty self-explanatory.



## Vectors

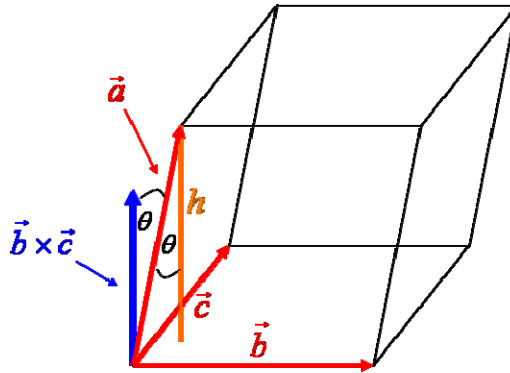
Let's look at an example of this. If  $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$  and  $\vec{b} = -\hat{i} + 2\hat{j} - 2\hat{k}$ , then the

cross product of these two vectors is  $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ -1 & 2 & -2 \end{vmatrix} = -14\hat{i} + 0\hat{j} + 7\hat{k} = -14\hat{i} + 7\hat{k}$ .

Hence, the area of the parallelogram with sides  $\vec{a}$  and  $\vec{b}$  is

$$\text{Area} = \|\vec{a} \times \vec{b}\| = \|-14\hat{i} + 7\hat{k}\| = \sqrt{245} = 7\sqrt{5} \approx 15.65.$$

Now for another application. Suppose we have a parallelepiped whose base is a parallelogram defined by vectors  $\vec{b}$  and  $\vec{c}$ , and the third defining side is vector  $\vec{a}$ .

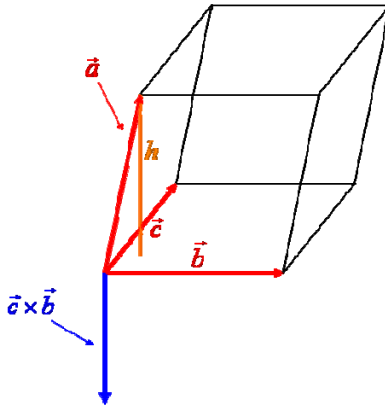


Then *Area of base*  $= \|\vec{b} \times \vec{c}\|$  and *height*  $= \|\vec{a}\| \cos \theta$ , where  $\theta$  is the acute angle between  $\vec{b} \times \vec{c}$  and  $\vec{a}$ . Then this suggests the following formula for the volume of the parallelepiped.

$$\text{Volume} = \text{Area of base} \times \text{height} = \|\vec{b} \times \vec{c}\| \cdot \|\vec{a}\| \cos \theta = (\vec{b} \times \vec{c}) \cdot \vec{a}$$

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However, notice that if we don't take our cross product in just the right order, then  $\theta$  will be an obtuse angle and  $\cos\theta$  will be negative.



The easy fix for this is to not worry about the order and simply take the absolute value of our final result in order to ensure that we get back a positive number for the volume. Hence,  $Volume = |(\vec{c} \times \vec{b}) \cdot \vec{a}| = |(\vec{b} \times \vec{c}) \cdot \vec{a}| = |\vec{a} \cdot (\vec{b} \times \vec{c})|$ . What may not be immediately apparent at this point, but is nonetheless easy for you to deduce is that

$|\vec{a} \cdot (\vec{b} \times \vec{c})| = \text{absolute value of } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ . Hence, the volume of the parallelepiped

defined by vectors  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$  is the absolute value of  $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ . Just do the

math, and you'll see that this is correct.

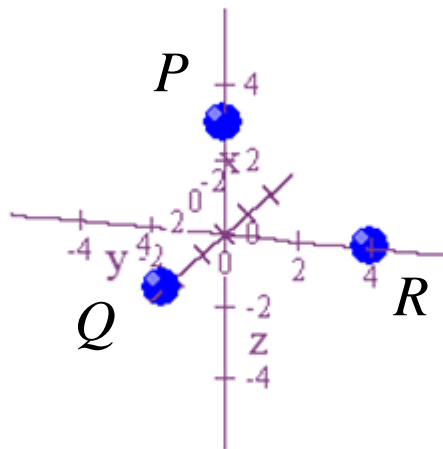
Now let's look at a quick example. Suppose our parallelepiped is defined by the vectors  $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ ,  $\vec{b} = -\hat{i} + 2\hat{j} - 2\hat{k}$ , and  $\vec{c} = -\hat{i} - 3\hat{j} + 4\hat{k}$ . Then the value of the

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determinant is  $\begin{vmatrix} 2 & 3 & 4 \\ -1 & 2 & -2 \\ -1 & -3 & 4 \end{vmatrix} = 42$ . Hence, the volume of the parallelepiped is

$|42| = 42$ . What could be simpler!

Now let's look at a really important application. Remember that we have a special interest in being able to construct planes because in three dimensions the tangent plane to a point on a surface is what corresponds to a tangent line in two dimensions, and we already know how important tangent lines are to calculus. Thus, let's suppose that we have three distinct points that define a plane such as  $P = (0, 0, 3)$ ,  $Q = (5, 0, 0)$ , and  $R = (0, 4, 0)$ .



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Then  $\overline{PQ} = \langle 5, 0, -3 \rangle$  and  $\overline{PR} = \langle 0, 4, -3 \rangle$  (notice that I've switched to an alternate notation in order to give you some practice with it). Also, we can construct parametric equations for line segments in order to add representations for these vectors to our graph. We'll parametrize the vector from  $P$  to  $Q$  as:

$$x = 0 + 5t = 5t$$

$$y = 0 + 0t = 0$$

$$z = 3 - 3t$$

$$0 \leq t \leq 1$$

And we can parametrize the vector from  $P$  to  $R$  as:

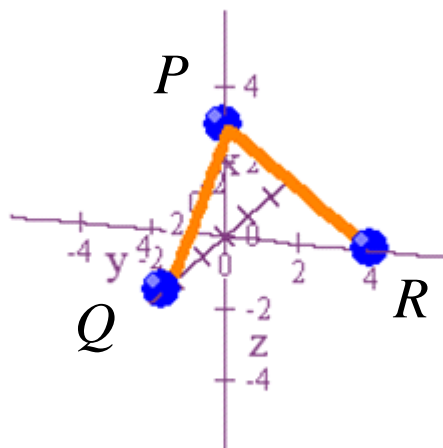
$$x = 0 + 0t = 0$$

$$y = 0 + 4t = 4t$$

$$z = 3 - 3t$$

$$0 \leq t \leq 1$$

And here is the picture we get,



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If we now take the cross product of  $\overline{PQ}$  and  $\overline{PR}$ , then we will have a vector perpendicular to the plane defined by  $\overline{PQ}$  and  $\overline{PR}$ .

$$\overline{PQ} \times \overline{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 0 & -3 \\ 0 & 4 & -3 \end{vmatrix} = 12\hat{i} + 15\hat{j} + 20\hat{k}$$

Parametric equations for this cross product vector starting at point  $P$  are:

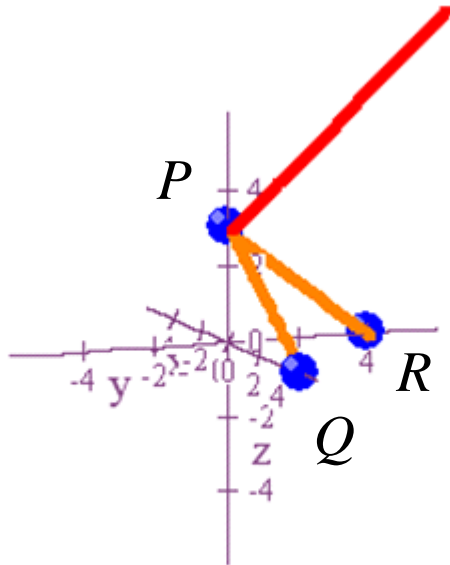
$$x = 0 + 12t = 12t$$

$$y = 0 + 15t = 15t$$

$$z = 3 + 20t$$

$$0 \leq t \leq 1$$

And here's what we get when we add this vector to the graph. We can certainly see that it looks perpendicular to the other two!



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Now suppose that we have some point  $X = (x, y, z)$  that lies in the plane defined by

$\overline{PQ}$  and  $\overline{PR}$ . Then the vector  $\overline{PX} = (x-0)\hat{i} + (y-0)\hat{j} + (z-3)\hat{k}$  lies in the plane, and it

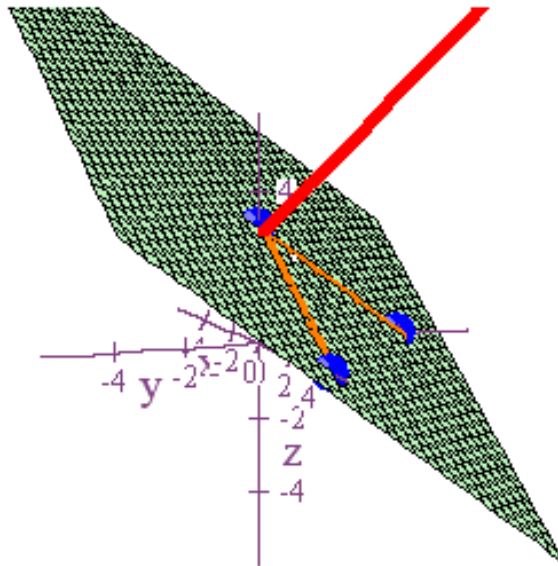
thus, must be perpendicular to the cross product  $\overline{PQ} \times \overline{PR} = 12\hat{i} + 15\hat{j} + 20\hat{k}$ . Hence,

$$(\overline{PQ} \times \overline{PR}) \cdot \overline{PX} = 0 \Rightarrow \langle 12, 15, 20 \rangle \cdot \langle x, y, z-3 \rangle = 0 \Rightarrow 12x + 15y + 20(z-3) = 0$$

$\Rightarrow 12x + 15y + 20z - 60 = 0$ . But this last equation is an equation for a plane, and if we

solve it for  $z$ , then we can rewrite it as  $z = -\frac{12}{20}x - \frac{15}{20}y + \frac{60}{20} = -\frac{3}{5}x - \frac{3}{4}y + 3$ . Let's

graph this last equation and add it to what we have so far.



Wonderful! We can easily see that we've taken our three original points and successfully found an equation for the plane they define as well as a vector which is perpendicular to this plane. Speaking of the latter, notice that the components of our

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vector  $\overline{PQ} \times \overline{PR} = 12\hat{i} + 15\hat{j} + 20\hat{k}$  appear also as coefficients of the equation for the plane,  $12x + 15y + 20z - 60 = 0$ . This is no accident. In fact, we can show that if we have an equation for a plane in the form  $Ax + By + Cz + D = 0$ , then the vector  $A\hat{i} + B\hat{j} + C\hat{k}$  is perpendicular to this plane. How do we do that? I'll show you!

Suppose  $Ax + By + Cz + D = 0$  is an equation for a plane. Then clearly

$Ax + By + Cz = -D$ . Now suppose that  $(a, b, c)$  is a point in the plane. Then it also follows that  $A \cdot a + B \cdot b + C \cdot c = -D$ . Additionally, if  $(x, y, z)$  is another point in our plane, then the displacement vector from  $(a, b, c)$  to  $(x, y, z)$  is  $\langle x - a, y - b, z - c \rangle$ .

Let's now compute the dot product between this last vector and the vector  $\langle A, B, C \rangle$ .

In particular, we get  $\langle A, B, C \rangle \cdot \langle x - a, y - b, z - c \rangle = A(x - a) + B(y - b) + C(z - c)$   
 $= Ax - A \cdot a + By - B \cdot b + Cz - C \cdot c = (Ax + By + Cz) - (A \cdot a + B \cdot b + C \cdot c) = -D + D = 0$ .

Therefore, the two vectors are perpendicular, and since the point  $(x, y, z)$  was chosen arbitrarily from the plane, it follows that  $\langle A, B, C \rangle$  is perpendicular to any vector in that plane, and hence,  $\langle A, B, C \rangle$  is perpendicular to the plane  $Ax + By + Cz + D = 0$ .

If we summarize a few things we know at this point, then we can say that if

$Ax + By + Cz + D = 0$  is any equation for a plane, then the coefficients  $A$ ,  $B$ , and  $C$  define a vector perpendicular to that plane. Recall that the word orthogonal means



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the same as perpendicular. We also express this same idea by saying that the vector  $\langle A, B, C \rangle$  is normal to the plane. Additionally, if our equation for the plane is written in the form  $z = Ax + By + C$ , then the coefficient  $A$  is the slope of the plane in the direction of the positive  $x$ -axis, and the coefficient  $B$  is the slope of the plane in the direction of the positive  $y$ -axis. Cool Stuff! And this will all be very important to us as we continue.

To wrap things up, we're now just going to give you a list of algebraic properties of vectors for future reference. We won't provide proofs of these properties, but they are all pretty straight forward if you just apply the definitions. Also, proving them is a fun thing to do at parties to amaze your friends and be a popular person!

If  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$  are vectors and  $c$  and  $d$  are scalars, then,

1.  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2.  $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3.  $\vec{a} + \vec{0} = \vec{a}$
4.  $\vec{a} + (-\vec{a}) = \vec{0}$
5.  $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
6.  $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
7.  $(cd)\vec{a} = c(d\vec{a})$
8.  $1 \cdot \vec{a} = \vec{a}$
9.  $0 \cdot \vec{a} = \vec{0}$
10.  $\vec{a} + (-1)\vec{b} = \vec{a} - \vec{b}$

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If  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$  are vectors and  $k$  is a scalar, then,

1.  $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$
2.  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3.  $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4.  $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k\vec{b})$
5.  $\vec{0} \cdot \vec{a} = 0$

If  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$  are vectors and  $k$  is a scalar, then,

1.  $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
2.  $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$
3.  $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4.  $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5.  $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6.  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$