

The Fundamental Theorem of Calculus



The bottom line is that if $f(x)$ is continuous on $[a,b]$ and if $F(x)$ is an antiderivative of $f(x)$, then ...

$$\int_a^b f(x) dx = F(b) - F(a)$$

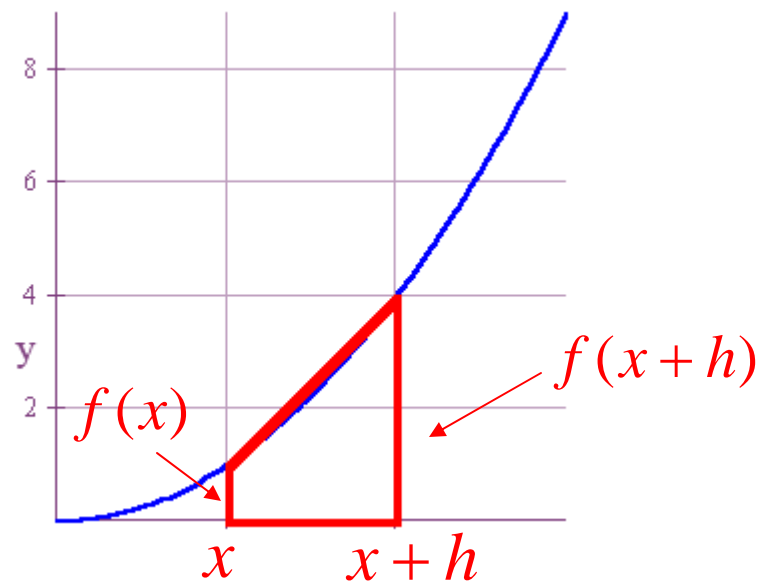
Here are some examples.

$$1. \int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

$$2. \int_1^2 e^x dx = e^x \Big|_1^2 = e^2 - e^1 = e^2 - e \approx 4.67077427$$

$$3. \int_1^2 \frac{1}{x} dx = \ln|x| \Big|_1^2 = \ln 2 - \ln 1 = \ln 2 \approx 0.6931471806$$

Now let's sketch out a proof of the fundamental theorem. We'll begin by noticing that the region below has an area that can be approximated by the area of a parallelogram.



$$\begin{aligned} \text{Area} &= \frac{\text{height}(b_1 + b_2)}{2} \\ &= \frac{h[f(x) + f(x+h)]}{2} \end{aligned}$$

Now consider $\int_a^b f(x) dx$, where $f(x)$ is continuous on $[a, b]$, and let

$$\begin{aligned}
 S(x) &= \int_a^x f(u) du. \quad \text{Then } S'(x) = \lim_{h \rightarrow 0} \frac{S(x+h) - S(x)}{h} = \lim_{h \rightarrow 0} \frac{\int_a^{x+h} f(u) du - \int_a^x f(u) du}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\int_a^{x+h} f(u) du + \int_x^a f(u) du}{h} = \lim_{h \rightarrow 0} \frac{\int_x^{x+h} f(u) du}{h} = \lim_{h \rightarrow 0} \frac{h[f(x) + f(x+h)]}{2h} \\
 &= \lim_{h \rightarrow 0} \frac{2hf(x)}{2h} = \lim_{h \rightarrow 0} \frac{hf(x)}{h} = f(x)
 \end{aligned}$$

Therefore, $S(x) = \int_a^x f(u) du$ is an antiderivative of $f(x)$.

Now let $F(x)$ be any antiderivative of $f(x)$. Then $F(x) - \int_a^x f(u) du = c$.

In particular, $c = F(a) - \int_a^a f(u) du = F(a) - 0 = F(a)$.

Thus, $F(b) - \int_a^b f(x) dx = F(a)$

$$\Rightarrow \int_a^b f(x) dx = F(b) - F(a)$$

The Fundamental Theorem of Calculus

Let $f(x)$ be continuous on $[a, b]$.

1. (The Derivative of the Integral)

If $S(x) = \int_a^x f(u) du$, then $S'(x) = f(x)$.

2. (The Integral of the Derivative)

If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x) dx = F(b) - F(a)$.

$$\int_a^b f(x) = F(b) - F(a)$$

ENJOY!

