

DIVERGENCE THEOREM IN THREE DIMENSIONS - ANSWERS

In each problem below you are given a volume V bounded by a surface S along with a vector field F . If N is an outward pointing unit normal vector, then the outward flux across the surface created by the vector field F is, by the Divergence (Gauss') Theorem, equal to $\iint_S \vec{F} \cdot N dS = \iiint_V \operatorname{div} \vec{F} dV = \iiint_V \nabla \cdot F dV$. Evaluate this integral for each problem below

1. V is the solid ball with surface S defined by $x^2 + y^2 + z^2 = 1$
 $F = x\hat{i} + y\hat{j} + z\hat{k}$

$$\iint_S \vec{F} \cdot N dS = \iiint_V \nabla \cdot F dV = \iiint_V (1+1+1) dV = 3 \iiint_V dV = 3 \cdot \frac{4\pi}{3} = 4\pi$$

2. V is the cube defined by $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$, and S is the surface of the cube
 $F = x^2\hat{i} + y\hat{j} + z\hat{k}$

$$\begin{aligned} \iint_S \vec{F} \cdot N dS &= \iiint_V \nabla \cdot F dV = \int_0^1 \int_0^1 \int_0^1 (2x+2) dx dy dz = \int_0^1 \int_0^1 x^2 + 2x \Big|_0^1 dy dz \\ &= \int_0^1 \int_0^1 3 dy dx = \int_0^1 3y \Big|_0^1 dz = \int_0^1 3 dz = 3z \Big|_0^1 = 3 \end{aligned}$$

3. V is the cylinder defined by $-1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, 0 \leq z \leq 4$,
 and S is the surface of the cylinder
 $F = y\hat{i} - x\hat{j}$

$$\iint_S \vec{F} \cdot N dS = \iiint_V \nabla \cdot F dV = \iiint_V 0 dV = 0$$

V is the solid bounded by the xy -plane and the hemisphere $z = \sqrt{4 - x^2 - y^2}$,

4. and S is the surface of V .

$$F = x^3 \hat{i} + y^3 \hat{j} + z^3 \hat{k}$$

$$\begin{aligned} \iint_S \vec{F} \cdot N dS &= \iiint_V \nabla \cdot F dV = \iiint_V (3x^2 + 3y^2 + 3z^2) dV \\ &= \iiint_V 3(x^2 + y^2 + z^2) dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 (3\rho^2) \cdot \rho^2 \sin \phi d\rho d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} 3\rho^4 \sin \phi d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi/2} \frac{3\rho^5}{5} \sin \phi \Big|_0^2 d\phi d\theta = \int_0^{2\pi} \int_0^{\pi/2} \frac{96}{5} \sin \phi d\phi d\theta \\ &= \int_0^{2\pi} -\frac{96}{5} \cos \phi \Big|_0^{\pi/2} d\theta = \int_0^{2\pi} \frac{96}{5} d\theta = \frac{96\theta}{5} \Big|_0^{2\pi} = \frac{192\pi}{5} \end{aligned}$$

V is the solid bounded by the portion in the first octant of the

5. cylinder defined by $x^2 + y^2 = 1$ as z varies from 0 to 1,
and S is the surface of V .

$$F = x^2 \hat{i}$$

$$\begin{aligned} \iint_S \vec{F} \cdot N dS &= \iiint_V \nabla \cdot F dV = \iiint_V 2x dV \\ &= \int_0^{\pi/2} \int_0^1 \int_0^1 2r \cos \theta \cdot r dz dr d\theta = \int_0^{\pi/2} \int_0^1 \int_0^1 2r^2 \cos \theta dz dr d\theta \\ &= \int_0^{\pi/2} \int_0^1 2r^2 \cos \theta \cdot z \Big|_0^1 dr d\theta = \int_0^{\pi/2} \int_0^1 2r^2 \cos \theta dr d\theta \\ &= \int_0^{\pi/2} \frac{2r^3}{3} \cos \theta \Big|_0^1 d\theta = \int_0^{\pi/2} \frac{2}{3} \cos \theta d\theta = \frac{2}{3} \sin \theta \Big|_0^{\pi/2} = \frac{2}{3} \end{aligned}$$