

STOKES' THEOREM IN THREE DIMENSIONS - ANSWERS

In each problem below you are given a surface S , defined by $z = f(x, y)$, over a region R , defined by the given limits on x and y . Let C_R be the boundary of the region R , oriented counterclockwise, and let C be the corresponding boundary on the surface S , also oriented counterclockwise. Then if F is a vector field and N is the upward pointing unit normal vector for the surface S , use the higher dimensional version of Stokes' Theorem, $\int_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } F \cdot N) dS$, to measure the circulation around the curve C that is caused by the vector field F .

Consider $z = f(x, y)$ as a level surface for a function $g = -f(x, y) + z$. Then $N = \frac{\nabla g}{\|\nabla g\|}$ will be an upward pointing unit normal to the surface $z = f(x, y)$, and we will have

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } F \cdot N) dS = \iint_R \left[(\nabla \times F) \cdot \frac{\nabla g}{\|\nabla g\|} \right] \cdot \|\nabla g\| dA = \iint_R [(\nabla \times F) \cdot \nabla g] dA.$$

$$S : z = -x^2 - y^2 + 4$$

1. $R : 0 \leq x \leq 1, 0 \leq y \leq 2$

$$F = z\hat{i} + x\hat{j} + y\hat{k}$$

$$g = x^2 + y^2 + z$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} + \hat{k}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R [(\nabla \times F) \cdot \nabla g] dA = \int_0^1 \int_0^2 [(\hat{i} + \hat{j} + \hat{k}) \cdot (2x\hat{i} + 2y\hat{j} + \hat{k})] dy dx$$

$$= \int_0^1 \int_0^2 (2x + 2y + 1) dy dx = \int_0^1 2xy + y^2 + y \Big|_0^2 dx = \int_0^1 (4x + 6) dx$$

$$= 2x^2 + 6x \Big|_0^1 = 2 + 6 = 8$$

$$S: z = y$$

$$2. \quad R: 0 \leq x \leq 1, 0 \leq y \leq 1-x$$

$$F = -3y^2 \hat{i} + 4z \hat{j} + 6x \hat{k}$$

$$g = -y + z$$

$$\nabla g = -\hat{j} + \hat{k}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3y^2 & 4z & 6x \end{vmatrix} = -4\hat{i} - 6\hat{j} + 6y\hat{k}$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_R [(\nabla \times F) \cdot \nabla g] dA = \int_0^1 \int_0^{1-x} [(-4\hat{i} - 6\hat{j} + 6y\hat{k}) \cdot (-\hat{j} + \hat{k})] dy dx \\ &= \int_0^1 \int_0^{1-x} (6 + 6y) dy dx = \int_0^1 6y + 3y^2 \Big|_0^{1-x} dx = \int_0^1 [6(1-x) + 3(1-x)^2] dx \\ &= \int_0^1 (6 - 6x + 3 - 6x + 3x^2) dx = \int_0^1 (3x^2 - 12x + 9) dx \\ &= x^3 - 6x^2 + 9x \Big|_0^1 = 1 - 6 + 9 = 4 \end{aligned}$$

$$S: z = x^2 - y^2$$

$$3. \quad R: -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$$

$$F = x^2 \hat{i} + y^2 \hat{j} + z^2 \hat{k}$$

$$g = -x^2 + y^2 + z$$

$$\nabla g = -2x\hat{i} + 2y\hat{j} + \hat{k}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_R [(\nabla \times F) \cdot \nabla g] dA = \iint_R [(0\hat{i} + 0\hat{j} + 0\hat{k}) \cdot (-2x\hat{i} + 2y\hat{j} + \hat{k})] dA \\ &= \iint_R 0 dA = 0 \end{aligned}$$

$$S: z = 1 - x^2 - y^2$$

$$4. \quad R: -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$$

$$F = z\hat{i} + x\hat{j} + y\hat{k}$$

(NOTE: On this problem, let C_R and C be oriented clockwise. This means that your unit normal N will be pointing downward instead of upward.)

$$g = 1 - x^2 - y^2 - z$$

$$\nabla g = -2x\hat{i} - 2y\hat{j} - \hat{k}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_R [(\nabla \times F) \cdot \nabla g] dA = \iint_R [(\hat{i} + \hat{j} + \hat{k}) \cdot (-2x\hat{i} - 2y\hat{j} - \hat{k})] dA \\ &= \iint_R (-2x - 2y - 1) dA = \int_0^{2\pi} \int_0^1 (-2r \cos \theta - 2r \sin \theta - 1) \cdot r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (-2r^2 \cos \theta - 2r^2 \sin \theta - r) dr d\theta = \int_0^{2\pi} \left[-\frac{2r^3}{3} \cos \theta - \frac{2r^3}{3} \sin \theta - \frac{r^2}{2} \right]_0^1 d\theta \\ &= \int_0^{2\pi} \left(-\frac{2}{3} \cos \theta - \frac{2}{3} \sin \theta - \frac{1}{2} \right) d\theta = \left[-\frac{2}{3} \sin \theta + \frac{2}{3} \cos \theta - \frac{\theta}{2} \right]_0^{2\pi} = -\pi \end{aligned}$$

$$S: z = \frac{1}{2}\sqrt{1-x^2-y^2}$$

5. $R: -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$

$$F = x^2 \hat{i} + y^2 \hat{j} + z^2 \tan xy \hat{k}$$

(HINT: Use a simpler surface with the same bounding curve.)

Use $z = 0$

$$g = z$$

$$\nabla g = \hat{k}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \tan xy \end{vmatrix} = (xz^2 \sec^2 xy) \hat{i} - (yz^2 \sec^2 xy) \hat{j}$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_R [(\nabla \times F) \cdot \nabla g] dA = \iint_R [(xz^2 \sec^2 xy) \hat{i} - (yz^2 \sec^2 xy) \hat{j}] \cdot (\hat{k}) dA \\ &= \iint_R 0 dA = 0 \end{aligned}$$