

## CURVATURE - ANSWERS

For each of the following curves, find the curvature at the indicated value for  $t$ . Also, state the radius of the best fitting circle at this point (circle radius =  $r = \frac{1}{\kappa}$ ). If  $\kappa = 0$ , then state that circle radius =  $\infty$ .

1.  $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j}$ ,  $0 \leq t \leq 2\pi$ ,  $t = \frac{\pi}{4}$

$$\vec{r}'(t) = -\sin(t)\hat{i} + \cos(t)\hat{j}$$

$$\|\vec{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$

$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = -\sin(t)\hat{i} + \cos(t)\hat{j}$$

$$T'(t) = -\cos(t)\hat{i} - \sin(t)\hat{j}$$

$$\kappa = \frac{\|T'(t)\|}{\|\vec{r}'(t)\|} = 1$$

$$\kappa\left(\frac{\pi}{4}\right) = 1$$

circle radius = 1

2.  $\vec{r}(t) = 2\cos(t)\hat{i} - 2\sin(t)\hat{j}$ ,  $0 \leq t \leq 2\pi$ ,  $t = \frac{5\pi}{4}$

$$\vec{r}'(t) = -2\sin(t)\hat{i} - 2\cos(t)\hat{j}$$

$$\|\vec{r}'(t)\| = \sqrt{(-2\sin t)^2 + (-2\cos t)^2} = 2$$

$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = -\sin(t)\hat{i} - \cos(t)\hat{j}$$

$$T'(t) = -\cos(t)\hat{i} + \sin(t)\hat{j}$$

$$\kappa = \frac{\|T'(t)\|}{\|\vec{r}'(t)\|} = \frac{1}{2}$$

$$\kappa\left(\frac{5\pi}{4}\right) = \frac{1}{2}$$

$$\vec{r}\left(\frac{\pi}{4}\right) = \sqrt{2}\hat{i} - \sqrt{2}\hat{j}$$

circle radius = 2

3.  $\vec{r}(t) = (2 + 3t)\hat{i} + (1 + 4t)\hat{j}$ ,  $0 \leq t \leq 2$ ,  $t = 1$

$$\vec{r}'(t) = 3\hat{i} + 4\hat{j}$$

$$\|\vec{r}'(t)\| = \sqrt{3^2 + 4^2} = 5$$

$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$$

$$T'(t) = 0\hat{i} + 0\hat{j} = \vec{0}$$

$$\kappa = \frac{\|T'(t)\|}{\|\vec{r}'(t)\|} = 0$$

$$\kappa(1) = 0$$

$$\text{circle radius} = \infty$$

4.  $\vec{r}(t) = t\hat{i} + t^2\hat{j}$ ,  $-2 \leq t \leq 2$ ,  $t = 1$

$$\vec{r}'(t) = \hat{i} + (2t)\hat{j}$$

$$\|\vec{r}'(t)\| = \sqrt{1 + 4t^2}$$

$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{1}{\sqrt{1 + 4t^2}}\hat{i} + \frac{2t}{\sqrt{1 + 4t^2}}\hat{j}$$

$$T'(t) = \frac{-4t}{(1 + 4t^2)^{3/2}}\hat{i} + \frac{2}{(1 + 4t^2)^{3/2}}\hat{j}$$

$$T'(1) = \frac{-4}{5^{3/2}}\hat{i} + \frac{2}{5^{3/2}}\hat{j}$$

$$\|T'(1)\| = \sqrt{\frac{16}{5^3} + \frac{4}{5^3}} = \sqrt{\frac{20}{5^3}} = \sqrt{\frac{4}{25}} = \frac{2}{5}$$

$$\vec{r}'(1) = \hat{i} + 2\hat{j}$$

$$\|\vec{r}'(1)\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\kappa(1) = \frac{\|T'(1)\|}{\|\vec{r}'(1)\|} = \frac{2/5}{\sqrt{5}} = \frac{2}{5\sqrt{5}}$$

$$\text{circle radius} = \frac{5\sqrt{5}}{2}$$

5.  $\vec{r}(t) = t\hat{i} + t^2\hat{j}$ ,  $-2 \leq t \leq 2$ ,  $t = 0$

$$\vec{r}'(t) = \hat{i} + (2t)\hat{j}$$

$$\vec{r}'(0) = \hat{i}$$

$$\|\vec{r}'(0)\| = \|\hat{i}\| = 1$$

$$T'(t) = \frac{-4t}{(1+4t^2)^{3/2}}\hat{i} + \frac{2}{(1+4t^2)^{3/2}}\hat{j}$$

$$T'(0) = 2\hat{j}$$

$$\|T'(0)\| = \|2\hat{j}\| = 2$$

$$\kappa(0) = \frac{\|T'(0)\|}{\|\vec{r}'(0)\|} = \frac{2}{1} = 2$$

$$\text{circle radius} = \frac{1}{2}$$

6.  $\vec{r}(t) = \sin t\hat{i} + t\hat{j}$ ,  $0 \leq t \leq 2\pi$ ,  $t = \pi$

$$\vec{r}'(t) = \cos(t)\hat{i} + \hat{j}$$

$$\|\vec{r}'(t)\| = \sqrt{1 + \cos^2 t}$$

$$\|\vec{r}'(\pi)\| = \sqrt{1 + \cos^2 \pi} = \sqrt{2}$$

$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\cos t}{\sqrt{1 + \cos^2 t}}\hat{i} + \frac{1}{\sqrt{1 + \cos^2 t}}\hat{j}$$

$$T'(t) = \frac{\sqrt{1 + \cos^2 t}(-\sin t) - \cos t \cdot \frac{1}{2}(1 + \cos^2 t)^{-1/2} \cdot 2\cos t(-\sin t)}{1 + \cos^2 t}\hat{i}$$

$$- \frac{1}{2}(1 + \cos^2 t)^{-3/2} \cdot 2\cos t(-\sin t)\hat{j}$$

$$T'(\pi) = 0\hat{i} + 0\hat{j} = \vec{0}$$

$$\|T'(\pi)\| = 0$$

$$\kappa(\pi) = \frac{\|T'(\pi)\|}{\|\vec{r}'(\pi)\|} = \frac{0}{\sqrt{2}} = 0$$

$$\text{circle radius} = \infty$$