CYLINDRICAL INTEGRALS - ANSWERS

For each problem below, set up and evaluate a triple integral in cylindrical coordinates.

1. Use a triple integral in cylindrical coordinates to find the volume of a cylinder of height *H* and radius *R*.

$$\int_0^{2\pi}\int_0^{R}\int_0^{H} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{R} rz\Big|_0^{H}\,dr\,d\theta = \int_0^{2\pi}\int_0^{R} rH\,dr\,d\theta = \int_0^{2\pi}\frac{r^2H}{2}\Big|_0^{R}\,d\theta$$

$$= \int_0^{2\pi}\frac{R^2H}{2}\,d\theta = \frac{R^2H}{2}\theta\Big|_0^{2\pi} = \pi R^2 H$$

2. Let *V* be a sphere with center at the origin and radius $= R$. Find the volume of *V*.

$$2\int_0^{2\pi}\int_0^{R}\int_0^{\sqrt{R^2-r^2}} r\,dz\,dr\,d\theta = 2\int_0^{2\pi}\int_0^{R} rz\Big|_0^{\sqrt{R^2-r^2}}\,dr\,d\theta = 2\int_0^{2\pi}\int_0^{R} r\sqrt{R^2-r^2}\,dr\,d\theta$$

$$= -\int_0^{2\pi}\int_{R^2}^{0} u^{1/2}\,du\,d\theta = \int_0^{2\pi}\int_0^{R^2} u^{1/2}\,du\,d\theta = \int_0^{2\pi}\frac{2u^{3/2}}{3}\Big|_0^{R^2}\,d\theta = \int_0^{2\pi}\frac{2R^3}{3}\,d\theta$$

$$= \frac{2R^3}{3}\theta\Big|_0^{2\pi} = \frac{4}{3}\pi R^3$$

3. Find the volume of the solid bounded by the sphere $x^2+y^2+z^2=1$ and the cone $z=\sqrt{x^2+y^2}$.

$$\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_r^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} rz\Big|_r^{\sqrt{1-r^2}}\,dr\,d\theta = \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\left(r\sqrt{1-r^2}-r^2\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{-(1-r^2)^{3/2}}{3}-\frac{r^3}{3}\right)\Bigg|_0^{\frac{1}{\sqrt{2}}}\,d\theta = \int_0^{2\pi}\left(\frac{-\left(1-\frac{1}{2}\right)^{3/2}}{3}-\frac{\frac{1}{2^{3/2}}}{3}+\frac{1}{3}\right)d\theta$$

$$= \int_0^{2\pi}\left(-\frac{1}{3\sqrt{2}}+\frac{1}{3}\right)d\theta = \left(\frac{1}{3}-\frac{1}{3\sqrt{2}}\right)2\pi$$

4. Find the volume of the solid bounded above by $z = -x^2 - y^2 + 2$ and below by the *xy*-plane.

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_0^{-r^2+2} r\,dzdrd\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} rz\Big|_0^{-r^2+2} d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} \left(-r^3+2r\right)drd\theta$$

$$= \int_0^{2\pi}\left(\frac{-r^4}{4}+r^2\right)\Bigg|_0^{\sqrt{2}} d\theta = \int_0^{2\pi}\left(-1+2\right)d\theta = \int_0^{2\pi} d\theta = 2\pi$$

5. Find the volume of the solid bounded above by $z = -x^2 - y^2 + 1$ and below by $z = x^2 + y^2 - 1$.

$$\int_0^{2\pi}\int_0^{1}\int_{r^2-1}^{-r^2+1} r\,dzdrd\theta = \int_0^{2\pi}\int_0^{1} rz\Big|_{r^2-1}^{-r^2+1} drd\theta = \int_0^{2\pi}\int_0^{1} \left(-r^3+r\right)-\left(r^3-r\right)drd\theta$$

$$= \int_0^{2\pi}\int_0^{1}\left(-2r^3+2r\right)drd\theta = \int_0^{2\pi}\left(\frac{-2r^4}{4}+r^2\right)\Bigg|_0^{1} = \int_0^{2\pi}\left(-\frac{1}{2}+1\right)d\theta$$

$$= \int_0^{2\pi}\frac{1}{2}d\theta = \frac{\theta}{2}\Big|_0^{2\pi} = \pi$$

6. Suppose you have a bead of radius 3 mm, and you drill a hole of radius 1 mm through the center. Find the volume that remains.

Using symmetry,

$$8\int_0^{\pi/2}\int_1^{3}\int_0^{\sqrt{9-r^2}} r\,dzdrd\theta = 8\int_0^{\pi/2}\int_1^{3} r\sqrt{9-r^2}\,drd\theta\ [set\,u = 9-r^2] = -4\int_0^{\pi/2}\int_8^{0} u^{1/2}\,dud\theta$$

$$= 4\int_0^{\pi/2}\int_0^{8} u^{1/2}\,dud\theta = 4\int_0^{\pi/2}\frac{2u^{3/2}}{3}\Bigg|_0^{8} d\theta = 4\int_0^{\pi/2}\frac{32\sqrt{2}}{3}d\theta = \frac{128\sqrt{2}}{3}\theta\Bigg|_0^{\pi/2} = \frac{64\sqrt{2}}{3}\pi$$