

POLAR INTEGRALS - ANSWERS

Do the following by changing to polar coordinates.

1. Find the area of one petal of the rose $r = \cos 2\theta$.

$$\begin{aligned} \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r dr d\theta &= 2 \int_0^{\pi/4} \int_0^{\cos 2\theta} r dr d\theta = 2 \int_0^{\pi/4} \left. \frac{r^2}{2} \right|_0^{\cos 2\theta} d\theta = \int_0^{\pi/4} \cos^2 2\theta d\theta \\ &= \int_0^{\pi/4} \frac{1 + \cos 4\theta}{2} d\theta = \left. \frac{\theta}{2} + \frac{\sin 4\theta}{8} \right|_0^{\pi/4} = \frac{\pi}{8} \end{aligned}$$

2. Prove that the area of a circle is πr^2 by evaluating $\iint_R dA$ where R is the disk $x^2 + y^2 \leq r^2$.

$$\iint_R dA = \int_0^{2\pi} \int_0^r r dr d\theta = \int_0^{2\pi} \left. \frac{r^2}{2} \right|_0^r d\theta = \int_0^{2\pi} \frac{r^2}{2} d\theta = \left. \frac{\theta r^2}{2} \right|_0^{2\pi} = \pi r^2$$

3. Evaluate $\iint_R \sqrt{x^2 + y^2} dA$ where R is the disk $x^2 + y^2 \leq 1$.

$$\begin{aligned} \iint_R \sqrt{x^2 + y^2} dA &= \int_0^{2\pi} \int_0^1 \sqrt{r^2} \cdot r dr d\theta = \int_0^{2\pi} \int_0^1 r^2 dr d\theta = \int_0^{2\pi} \left. \frac{r^3}{3} \right|_0^1 d\theta \\ &= \int_0^{2\pi} \frac{1}{3} d\theta = \left. \frac{\theta}{3} \right|_0^{2\pi} = \frac{2\pi}{3} \end{aligned}$$

4. Find the volume of the solid bounded above by $z = x^2 + y^2 + 1$ and below by the disk $x^2 + y^2 \leq 1$.

$$\begin{aligned} \iint_R (x^2 + y^2 + 1) dA &= \int_0^{2\pi} \int_0^1 (r^2 + 1) r dr d\theta = \int_0^{2\pi} \int_0^1 (r^3 + r) dr d\theta \\ &= \int_0^{2\pi} \left. \left(\frac{r^4}{4} + \frac{r^2}{2} \right) \right|_0^1 d\theta = \int_0^{2\pi} \frac{3}{4} d\theta = \left. \frac{3\theta}{4} \right|_0^{2\pi} = \frac{3\pi}{2} \end{aligned}$$

5. Find the surface area of the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy -plane

$$\begin{aligned}\text{surface area} &= \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA = \iint_R \sqrt{(-2x)^2 + (-2y)^2 + 1} dA \\ &= \iint_R \sqrt{4x^2 + 4y^2 + 1} dA = \iint_R \sqrt{4r^2 + 1} r dr d\theta = \int_0^{2\pi} \int_0^2 (4r^2 + 1)^{1/2} r dr d\theta \\ &= \frac{1}{8} \int_0^{2\pi} \int_1^{17} u^{1/2} du d\theta = \int_0^{2\pi} \frac{u^{3/2}}{12} \Big|_1^{17} d\theta = \int_0^{2\pi} \frac{17^{3/2} - 1}{12} d\theta = \frac{\pi}{6} (17^{3/2} - 1) \approx 36.18\end{aligned}$$