

Triple Integrals



Change in volume $= \Delta V = \Delta x \Delta y \Delta z$

$$dV = dx dy dz$$

Let $w = f(x, y, z)$

$$\iiint_V f(x, y, z) dV = \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \sum_{i, j, k} f(x_{ijk}, y_{ijk}, z_{ijk}) \Delta V$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x, y)}^{h_2(x, y)} f(x, y, z) dz dy dx$$

$$\text{Volume} = \iiint_V dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x, y)}^{h_2(x, y)} dz dy dx$$

EXAMPLE: Find the volume of the solid in the first octant that is bounded below by the xy -plane and above by $x + y + z = 1$.

$$0 \leq x \leq 1$$

$$0 \leq y \leq -x + 1$$

$$0 \leq z \leq 1 - x - y$$

$$\begin{aligned} \text{Volume} &= \iiint_V dV = \int_0^1 \int_0^{-x+1} \int_0^{1-x-y} dz dy dx = \int_0^1 \int_0^{-x+1} (1-x-y) dy dx \\ &= \int_0^1 \left(y - xy - \frac{y^2}{2} \right) \Big|_0^{-x+1} dx = \int_0^1 \left(-x+1 - x(-x+1) - \frac{(-x+1)^2}{2} \right) dx \\ &= \int_0^1 \left(-x+1 + x^2 - x - \frac{x^2 - 2x + 1}{2} \right) dx = \int_0^1 \left(\frac{x^2}{2} - x + \frac{1}{2} \right) dx \\ &= \left(\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{2} \right) \Big|_0^1 = \frac{1}{6} \end{aligned}$$

EXAMPLE: Let R be the solid region between the graphs of $z = -y^2$ and $z = x^2$ where $0 \leq x \leq 1$ and $0 \leq y \leq x$. Evaluate $\iiint_R (x+1) dV$.

$$0 \leq x \leq 1$$

$$0 \leq y \leq x$$

$$-y^2 \leq z \leq x^2$$

$$\begin{aligned} \iiint_V (x+1) dV &= \int_0^1 \int_0^x \int_{-y^2}^{x^2} (x+1) dz dy dx = \int_0^1 \int_0^x z(x+1) \Big|_{-y^2}^{x^2} dy dx \\ &= \int_0^1 \int_0^x (x^3 + x^2) + (xy^2 + y^2) dy dx = \int_0^1 \left(x^3 y + x^2 y + \frac{xy^3}{3} + \frac{y^3}{3} \right) \Big|_0^x dx \\ &= \int_0^1 \left(x^4 + x^3 + \frac{x^4}{3} + \frac{x^3}{3} \right) dx = \int_0^1 \left(\frac{4x^4}{3} + \frac{4x^3}{3} \right) dx = \left(\frac{4x^5}{15} + \frac{4x^4}{12} \right) \Big|_0^1 \\ &= \frac{4}{15} + \frac{4}{12} = \frac{16}{60} + \frac{20}{60} = \frac{36}{60} = \frac{6}{10} = \frac{3}{5} \end{aligned}$$