

SPHERICAL INTEGRALS - ANSWERS

For each problem below, set up and evaluate a triple integral in spherical coordinates.

1. Let V be a sphere with center at the origin and radius $= r$. Find the volume of V .

$$\begin{aligned} \iiint_V dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^r \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \frac{\rho^3}{3} \sin \varphi \Big|_0^r \, d\varphi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \frac{r^3}{3} \sin \varphi \, d\varphi \, d\theta = \int_0^{2\pi} -\frac{r^3}{3} \cos \varphi \Big|_0^{\pi} \, d\theta = \int_0^{2\pi} -\frac{r^3}{3} (\cos \pi - \cos 0) \, d\theta \\ &= \int_0^{2\pi} \frac{2r^3}{3} \, d\theta = \frac{2r^3}{3} \Big|_0^{2\pi} = \frac{4}{3} \pi r^3 \end{aligned}$$

2. Find the volume of the solid bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

$$\begin{aligned} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{\rho^3}{3} \sin \varphi \Big|_0^1 \, d\varphi \, d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{1}{3} \sin \varphi \, d\varphi \, d\theta \\ &= \int_0^{2\pi} -\frac{1}{3} \cos \varphi \Big|_0^{\frac{\pi}{4}} \, d\theta = \int_0^{2\pi} \left(-\frac{1}{3\sqrt{2}} + \frac{1}{3} \right) \, d\theta = \left(\frac{1}{3} - \frac{1}{3\sqrt{2}} \right) \theta \Big|_0^{2\pi} \\ &= \left(\frac{1}{3} - \frac{1}{3\sqrt{2}} \right) 2\pi \end{aligned}$$

3. Evaluate $\iiint_V \frac{1}{x^2 + y^2 + z^2} dV$ where V is the solid region between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$.

$$\begin{aligned} \iiint_V \frac{1}{x^2 + y^2 + z^2} dV &= \int_0^{2\pi} \int_0^{\pi} \int_2^3 \frac{1}{\rho^2} \rho^2 \sin \varphi d\rho d\varphi d\theta = \int_0^{2\pi} \int_0^{\pi} \sin \varphi d\rho d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \rho \sin \varphi \Big|_2^3 d\varphi d\theta = \int_0^{2\pi} \int_0^{\pi} \sin \varphi d\varphi d\theta = \int_0^{2\pi} -\cos \varphi \Big|_0^{\pi} d\theta = \int_0^{2\pi} 2 d\theta \\ &= 2\theta \Big|_0^{2\pi} = 4\pi \end{aligned}$$

4. Find the volume of the solid bounded above by $z=1$ and below by $z=\sqrt{x^2 + y^2}$.

$$\begin{aligned} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sec \varphi} \rho^2 \sin \varphi d\rho d\varphi d\theta &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{\rho^3}{3} \sin \varphi \Big|_0^{\sec \varphi} d\varphi d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{\sec^3 \varphi}{3} \sin \varphi d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{\sin \varphi}{3 \cos^3 \varphi} d\varphi d\theta = \int_0^{2\pi} \frac{1}{6 \cos^2 \varphi} \Big|_0^{\frac{\pi}{4}} d\theta = \int_0^{2\pi} \left(\frac{1}{3} - \frac{1}{6} \right) d\theta = \int_0^{2\pi} \frac{1}{6} d\theta \\ &= \frac{\theta}{6} \Big|_0^{2\pi} = \frac{\pi}{3} \end{aligned}$$

5. Find the volume of the solid region defined by $\rho = \sin \varphi$ where $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi$.

$$\begin{aligned} \text{Volume} &= \int_0^{2\pi} \int_0^{\pi} \int_0^{\sin \varphi} \rho^2 \sin \varphi d\rho d\varphi d\theta = \int_0^{2\pi} \int_0^{\pi} \frac{\rho^3}{3} \sin \varphi \Big|_0^{\sin \varphi} d\varphi d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} \sin^4 \varphi d\varphi d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} (\sin^2 \varphi)^2 d\varphi d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} \left(\frac{1 - \cos 2\varphi}{2} \right)^2 d\varphi d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} \frac{1 - 2\cos 2\varphi + \cos^2 2\varphi}{4} d\varphi d\theta \\ &= \frac{1}{12} \int_0^{2\pi} \int_0^{\pi} 1 - 2\cos 2\varphi + \frac{1 + \cos 4\varphi}{2} d\varphi d\theta = \frac{1}{12} \int_0^{2\pi} \int_0^{\pi} \left(\frac{3}{2} - 2\cos 2\varphi + \frac{\cos 4\varphi}{2} \right) d\varphi d\theta \\ &= \frac{1}{12} \int_0^{2\pi} \left(\frac{3\varphi}{2} - \sin 2\varphi + \frac{\sin 4\varphi}{8} \right) \Big|_0^{\pi} d\theta = \frac{\pi}{8} \int_0^{2\pi} d\theta = \frac{\pi\theta}{8} \Big|_0^{2\pi} = \frac{\pi^2}{4} \end{aligned}$$