The **<u>Fun</u>**da**<u>mental</u>** Theorem of Calculus

The bottom line is that if f(x) is continuous on [a,b] and if F(x) is an antiderivative of f(x), then …

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

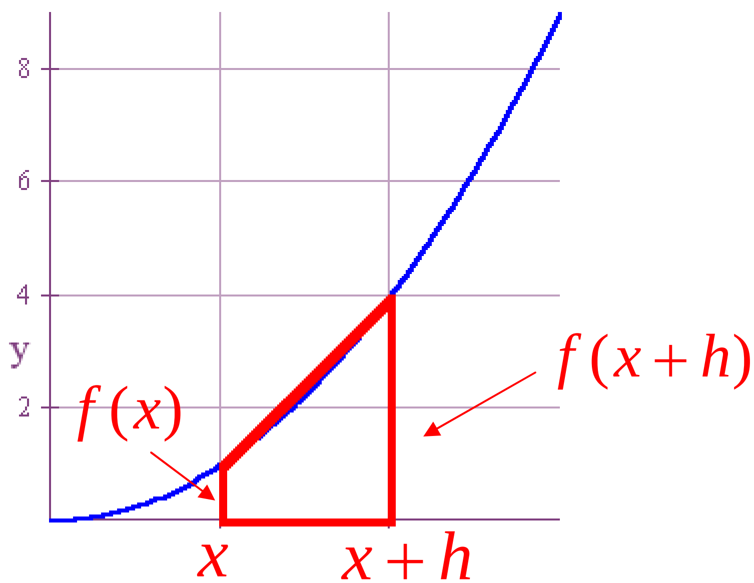
Here are some examples.

1. $$\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

2. $$\int_1^2 e^x\,dx = \left.e^x\right|_1^2 = e^2 - e^1 = e^2 - e \approx 4.67077427$$

3. $$\int_1^2 \frac{1}{x}\,dx = \left.\ln|x|\right|_1^2 = \ln 2 - \ln 1 = \ln 2 \approx 0.6931471806$$

Now let’s sketch out a proof of the fundamental theorem. We’ll begin by noticing that the region below has an area that can be approximated by the area of a trapezoid.



$$Area = \frac{height(b_1 + b_2)}{2}$$

$$= \frac{h[f(x) + f(x+h)]}{2}$$

Now consider $\int_a^b f(x)\,dx$, where $f(x)$ is continous on $[a,b]$, and let

$$S(x)=\int_a^x f(u)\,du. \text{ Then } S'(x)=\lim_{h\to 0}\frac{S(x+h)-S(x)}{h}=\lim_{h\to 0}\frac{\int_a^{x+h} f(u)\,du-\int_a^x f(u)\,du}{h}$$

$$=\lim_{h\to 0}\frac{\int_a^{x+h} f(u)\,du+\int_x^a f(u)\,du}{h}=\lim_{h\to 0}\frac{\int_x^{x+h} f(u)\,du}{h}=\lim_{h\to 0}\frac{\frac{h[f(x)+f(x+h)]}{2}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{2hf(x)}{2}}{h}=\lim_{h\to 0}\frac{hf(x)}{h}=f(x)$$

Therefore, $S(x)=\int_a^x f(u)\,du$ is an antiderivative of $f(x)$.

Now let $F(x)$ be any antiderivative of $f(x)$. Then $F(x) - \int_a^x f(u)\,du = c$.

In particular, $c = F(a) - \int_a^a f(u)\,du = F(a) - 0 = F(a)$.

Thus, $F(b) - \int_a^b f(x)\,dx = F(a)$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$$

The Fundamental Theorem of Calculus

Let $f(x)$ be continuous on $[a,b]$.

1. (The Derivative of the Integral)

   If $S(x)=\int_a^x f(u)\,du$, then $S'(x)=f(x)$.

2. (The Integral of the Derivative)

   If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)\,dx=F(b)-F(a)$.

Examples:

$$\int_0^4 x^2 \, dx = ?$$

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$$\int_0^4 x^2\,dx = \left.\frac{x^3}{3}\right|_0^4 = \frac{4^3}{3} - \frac{0^3}{3} = \frac{64}{3} = 21\frac{1}{3}$$

Examples:

$$\int_0^1 (x^3 + 1)\, dx = ?$$

Examples:

$$\int_0^1 (x^3+1)\,dx = ?$$

$$\int_0^1 (x^3+1)\,dx = \left.\frac{x^4}{4}+x\right|_0^1 = \left(\frac{1^4}{4}+1\right)-\left(\frac{0^4}{4}+0\right)=\frac{5}{4}=1.25$$

Examples:

$$\int_0^3 (-x-1)\,dx = ?$$

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$$\int_0^3 (-x-1)\,dx = ?$$

$$\int_0^3 (-x-1)\,dx = -\frac{x^2}{2} - x\Bigg|_0^3 = \left(-\frac{3^2}{2} - 3\right) - \left(-\frac{0^2}{2} - 0\right) = -\frac{15}{2} = -7.5$$

Examples:

$$\int_{-1}^{3} (x^3 - 3x)\,dx = ?$$

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$$\int_{-1}^{3} (x^3 - 3x)\,dx = ?$$

$$\int_{-1}^{3} (x^3 - 3x)\,dx = \left.\frac{x^4}{4} - \frac{3x^2}{2}\right|_{-1}^{3} = \left(\frac{81}{4} - \frac{27}{2}\right) - \left(\frac{1}{4} - \frac{3}{2}\right)$$

$$= \frac{27}{4} + \frac{5}{4} = \frac{32}{4} = 8$$

Examples:

$$\int_0^2 x^3 \, dx = ?$$

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$$\int_0^2 x^3\,dx = \left.\frac{x^4}{4}\right|_0^2 = \frac{16}{4} - \frac{0}{4} = 4$$

$$\int_a^b f(x) = F(b) - F(a)$$

ENJOY!