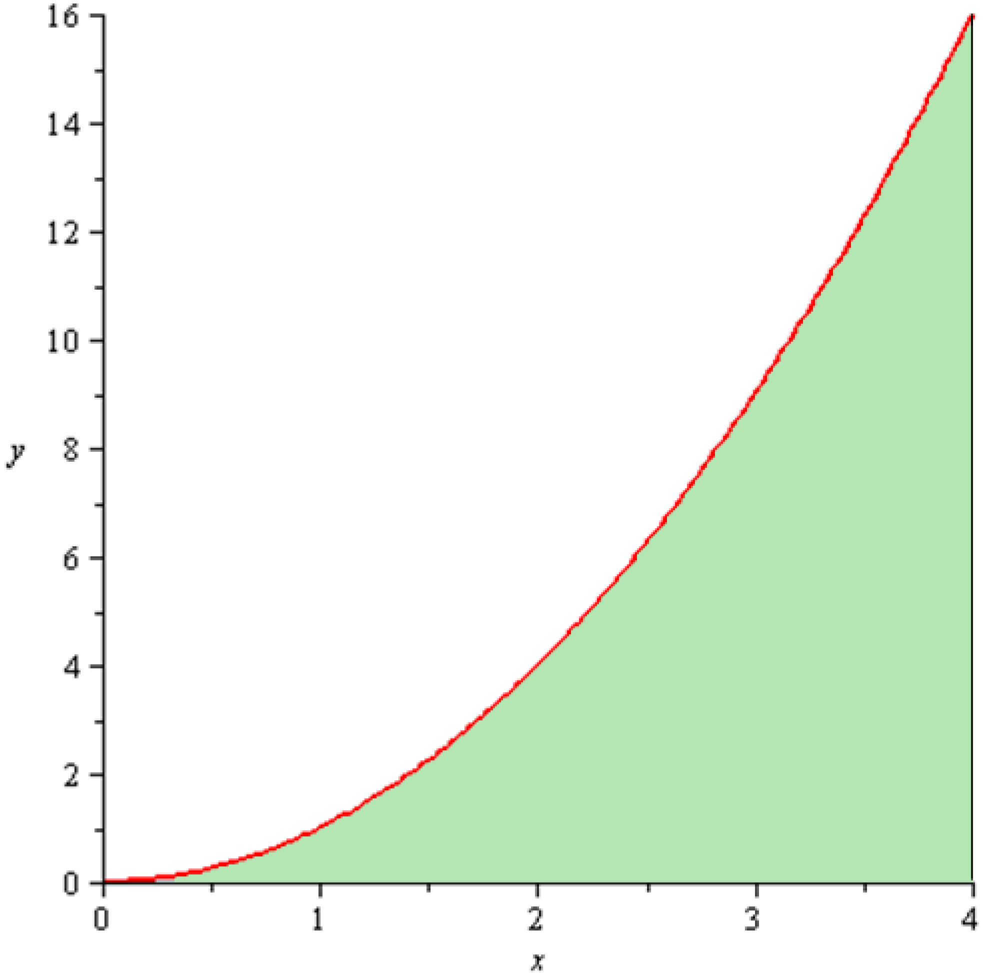
INTRODUCTION TO THE DEFINITE INTEGRAL

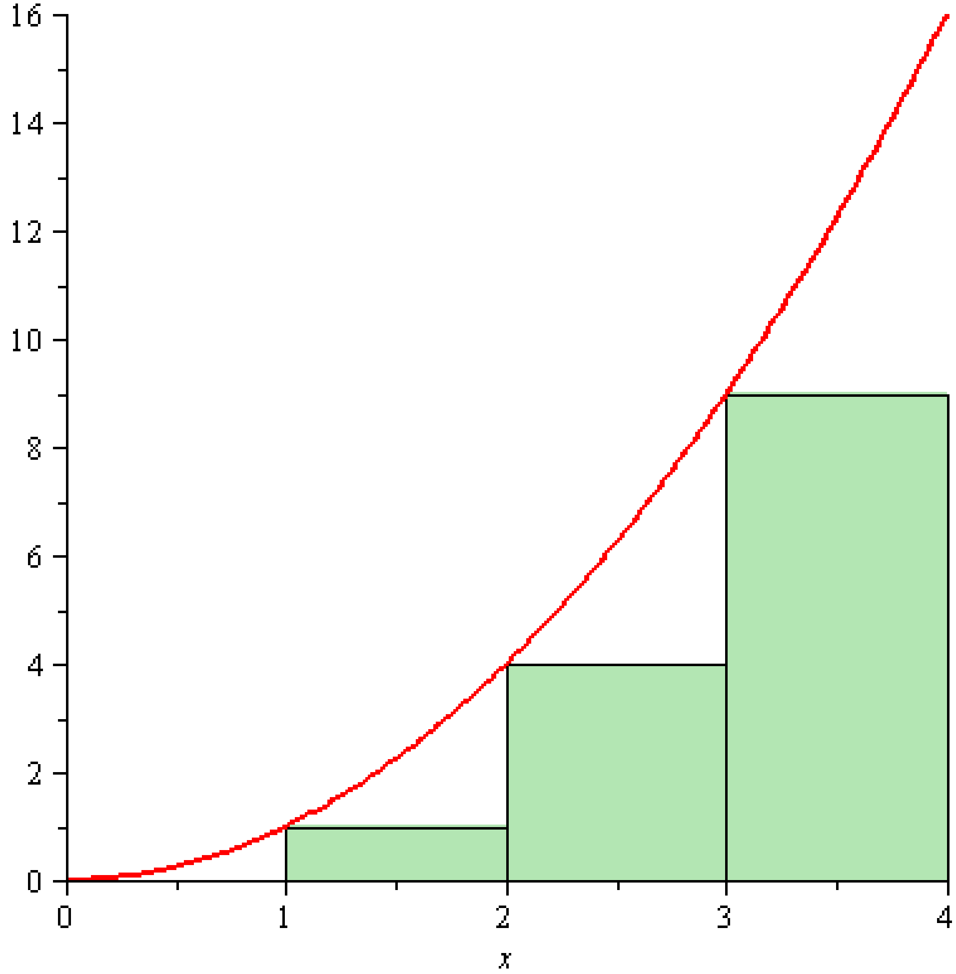
$$\int_a^b f(x)\,dx = ?$$

An interesting question to ask is, “What is the area under the graph of a given function?”



$$f(x) = x^2$$

One way to approximate this is by drawing rectangles and using the areas of the rectangles as an approximation of total area under the curve.

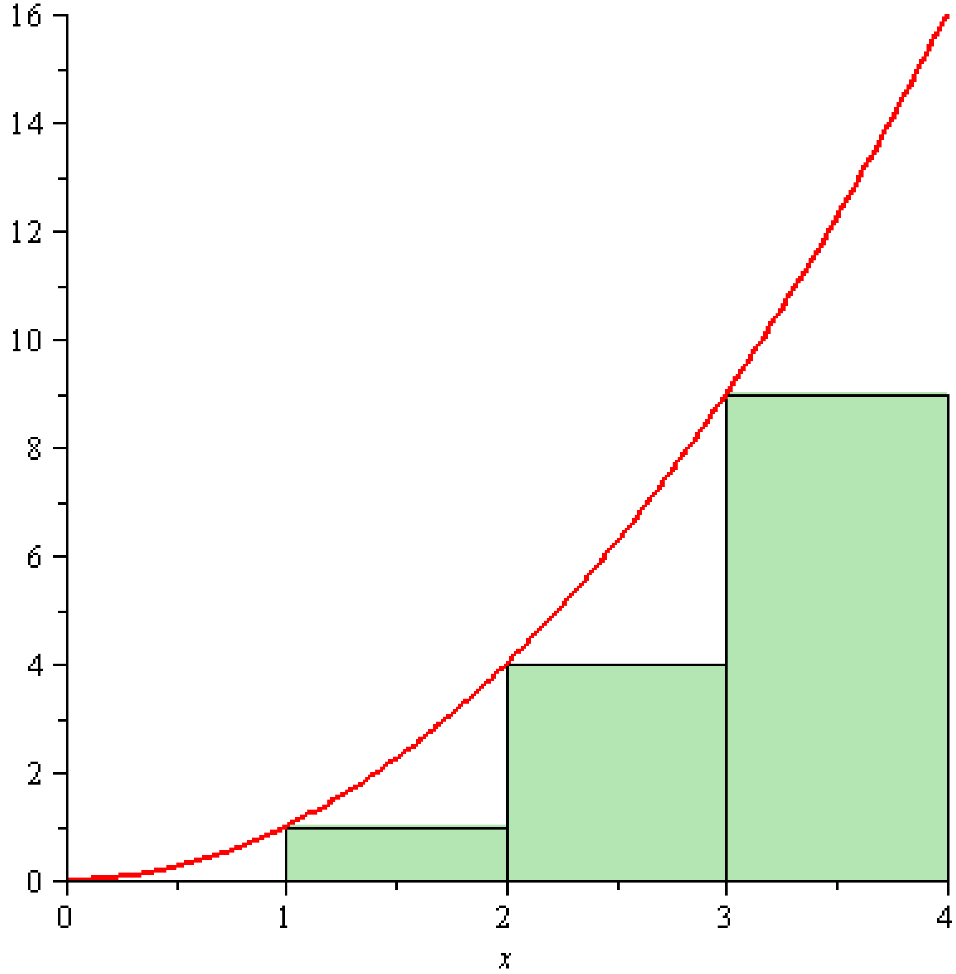


$f(x) = x^2$

In the drawing below, the get the width of each rectangle by dividing the length of the interval by the number of rectangles we want.

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{4-0}{4} = 1$$

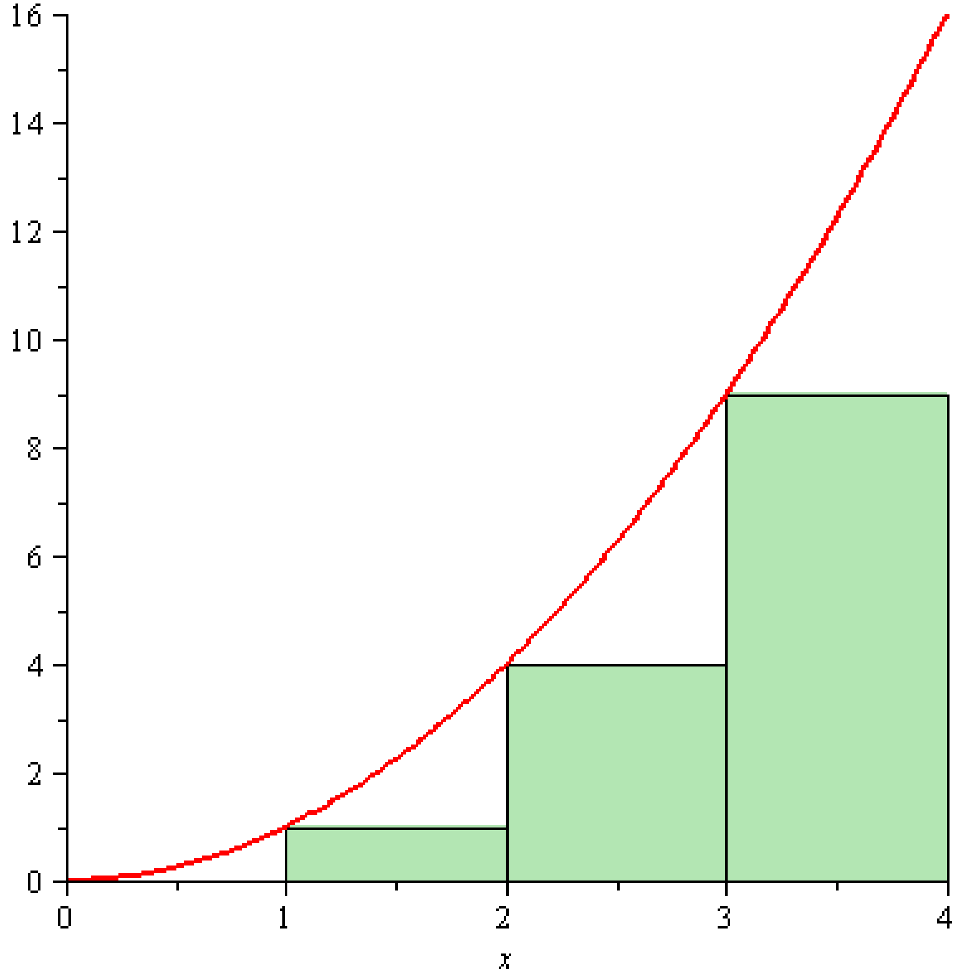


$$f(x) = x^2$$

To get the height of each rectangle, we evaluated our function at the left endpoint of each subinterval. This method results in a left sum or left riemann sum.

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{4-0}{4} = 1$$



$$f(x) = x^2$$

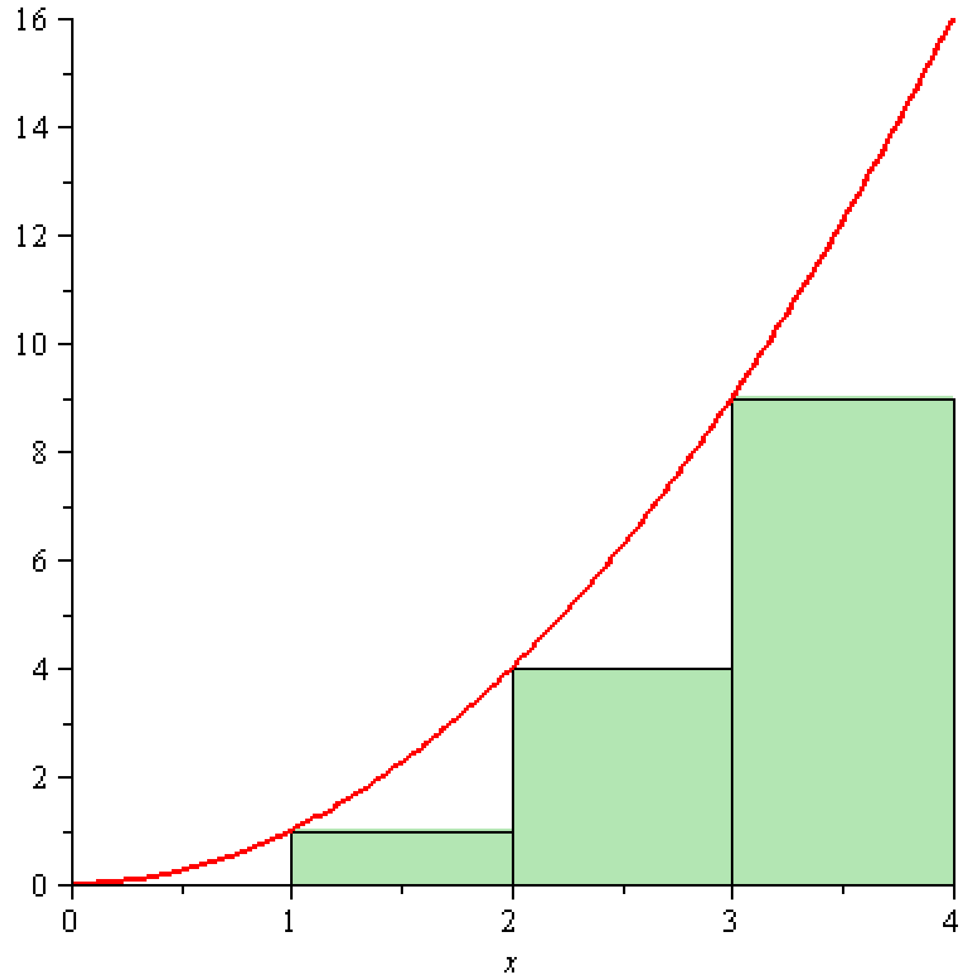
And here is the result.

$$\text{Area} \approx f(0)\cdot\Delta x + f(1)\cdot\Delta x + f(2)\cdot\Delta x + f(3)\cdot\Delta x$$

$$= 0\cdot 1 + 1\cdot 1 + 4\cdot 1 + 9\cdot 1 = 14$$

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{4-0}{4} = 1$$

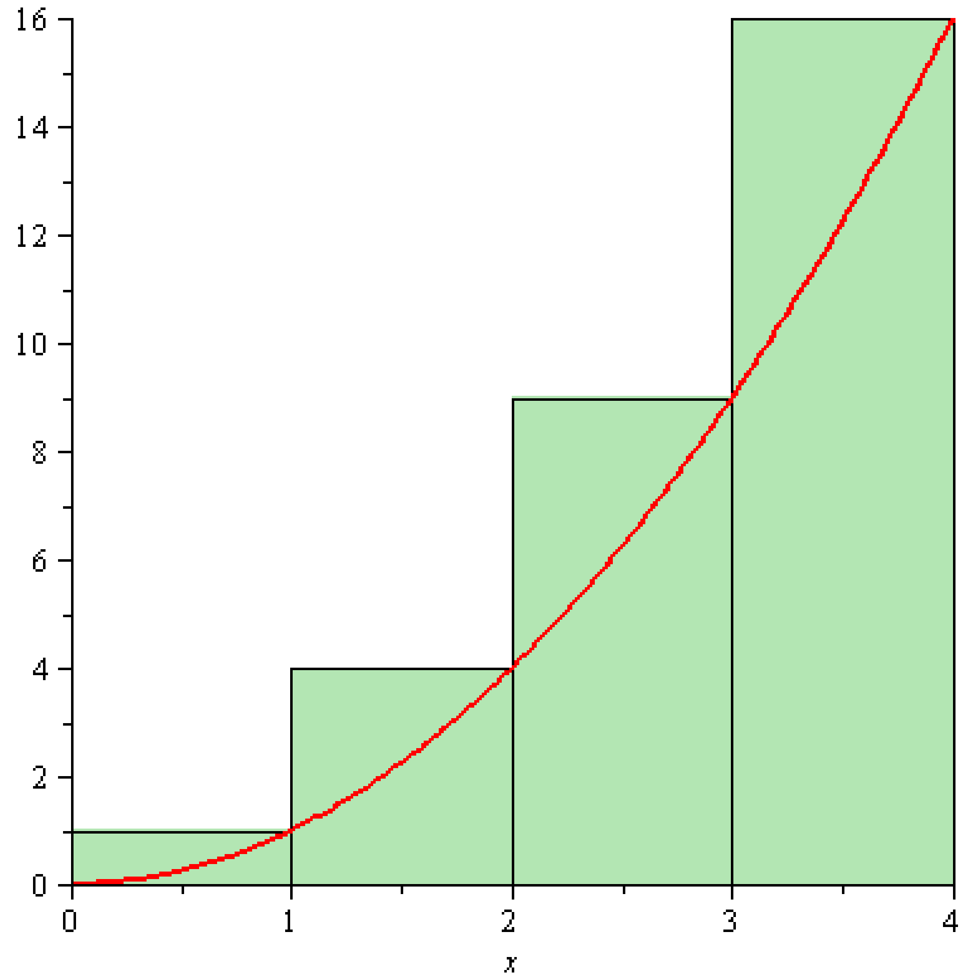


$$f(x) = x^2$$

Using the right endpoint of each interval gives a different approximation that we'll call a right sum.

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{4-0}{4} = 1$$



$$f(x) = x^2$$

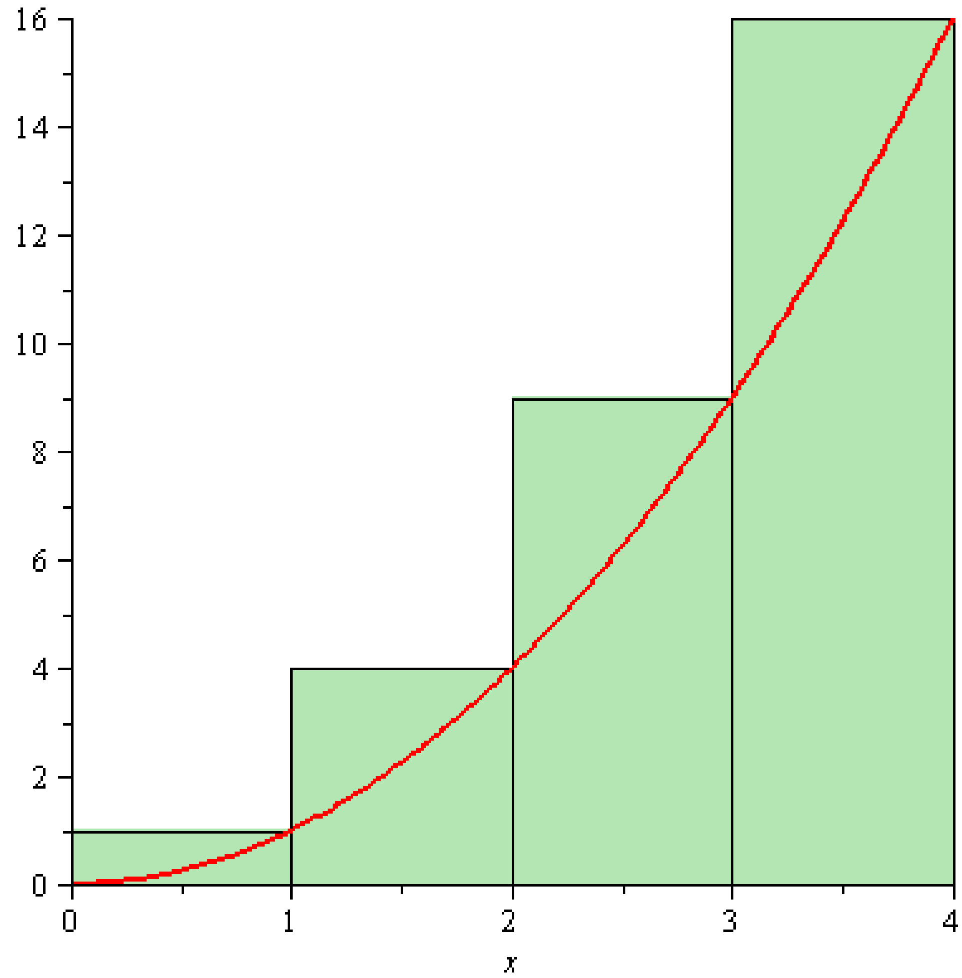
And here's the right sum approximation.

$$\text{Area} \approx f(1)\cdot\Delta x + f(2)\cdot\Delta x + f(3)\cdot\Delta x + f(4)\cdot\Delta x$$

$$= 1\cdot 1 + 4\cdot 1 + 9\cdot 1 + 16\cdot 1 = 30$$

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{4-0}{4} = 1$$

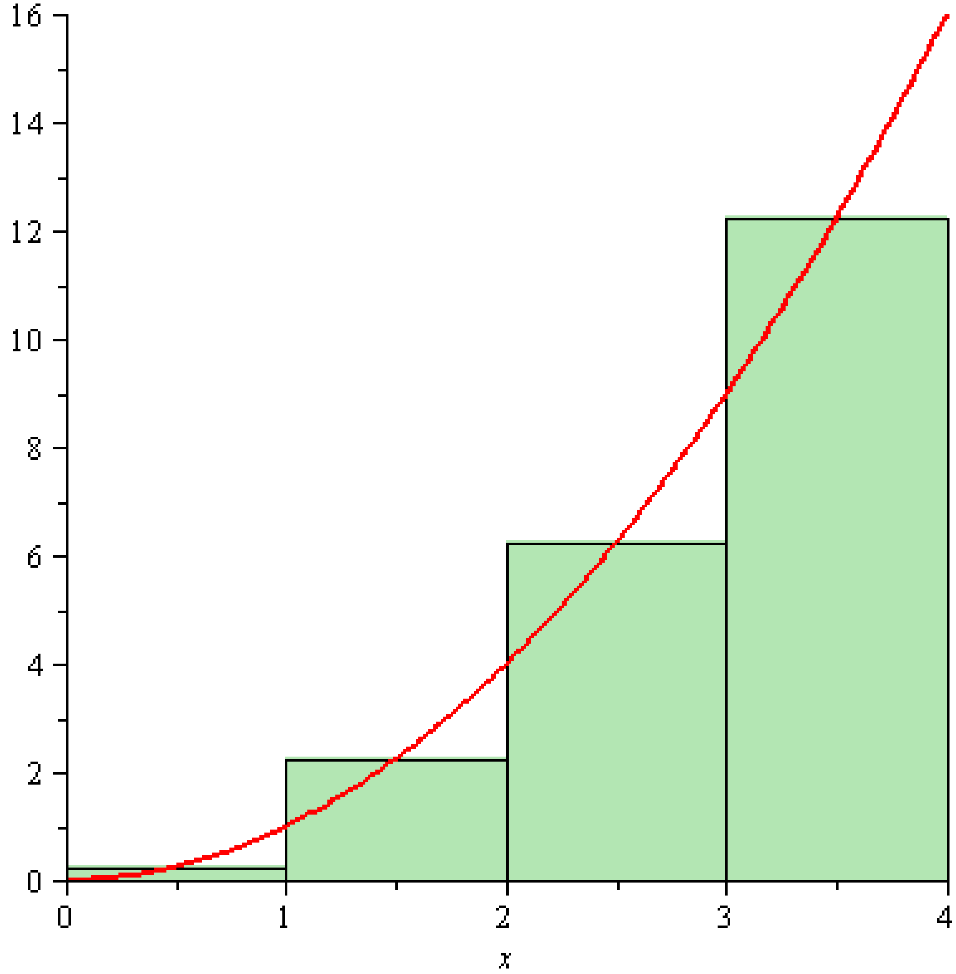


$$f(x) = x^2$$

If we use the midpoint of each interval to find the height of a rectangle, we'll call that a middle sum.

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{4-0}{4} = 1$$



$$f(x) = x^2$$

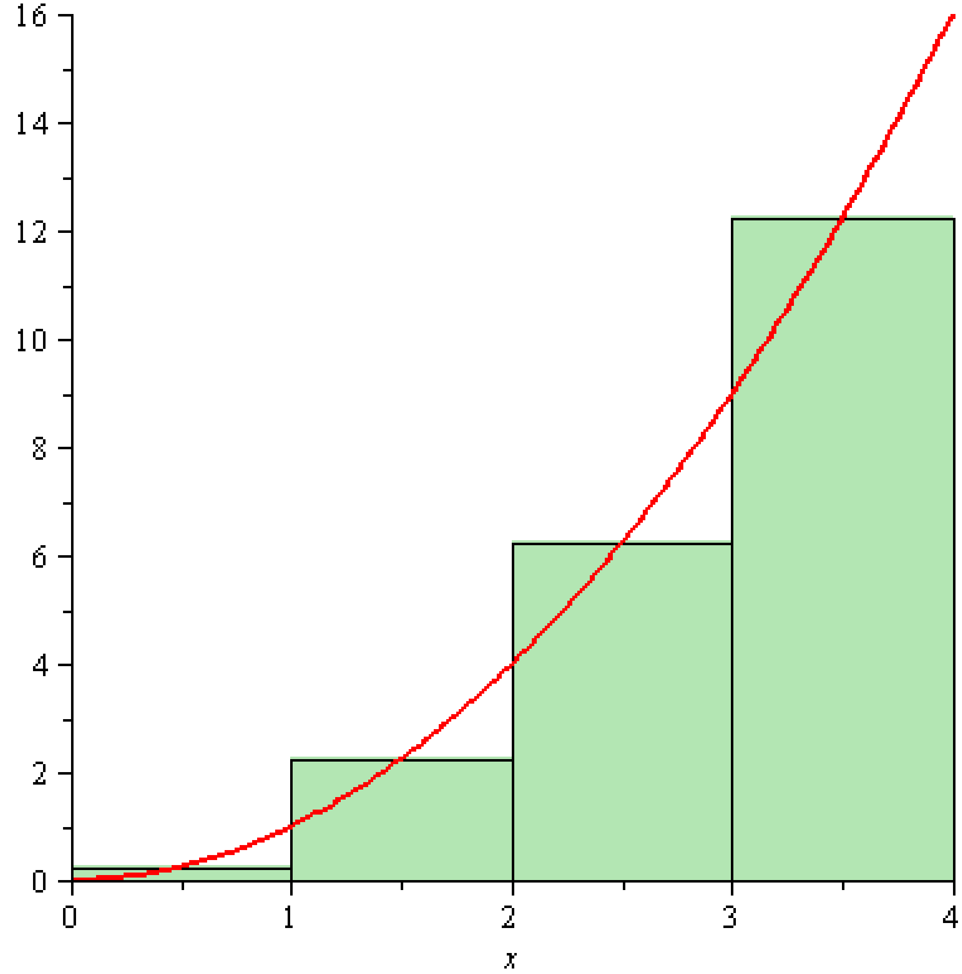
And here's the area estimate from the middle sum.

$$\text{Area} \approx f(.5)\cdot\Delta x + f(1.5)\cdot\Delta x + f(2.5)\cdot\Delta x + f(3.5)\cdot\Delta x$$

$$= .5^2 \cdot 1 + 1.5^2 \cdot 1 + 2.5^2 \cdot 1 + 3.5^2 \cdot 1 = 21$$

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{4-0}{4} = 1$$



$$f(x) = x^2$$

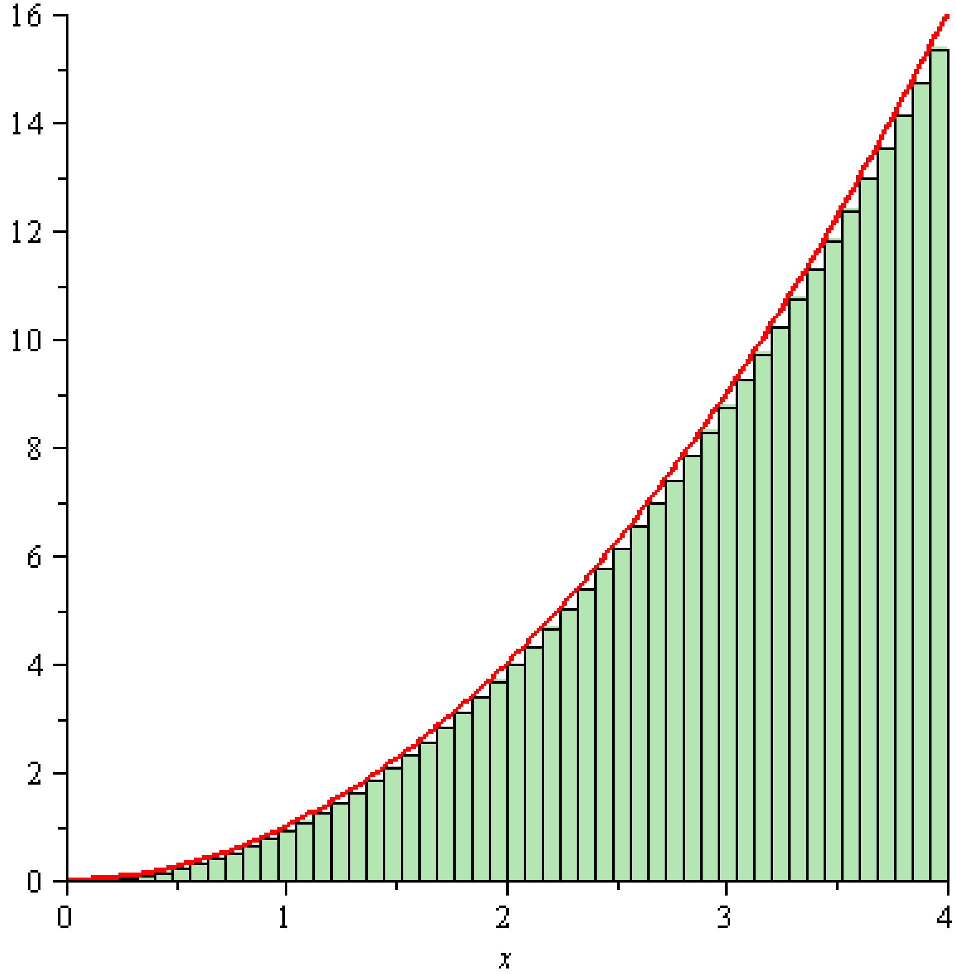
For each type of riemann sum, we can refine our area estimate by just using more rectangles.

Area ≈ 20.6976

$$\Delta x = \frac{b-a}{n}$$

left sum

$n = 50$



$f(x) = x^2$

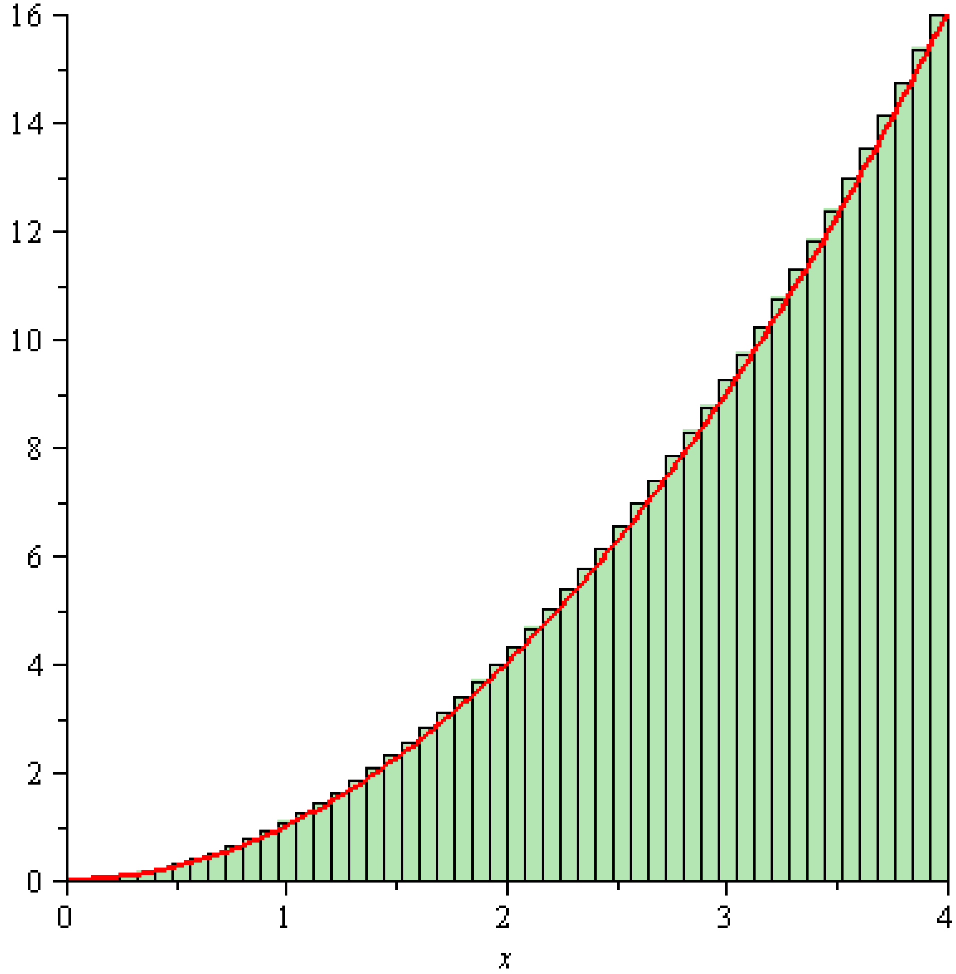
Here is the estimate for the right riemann sum.

$\text{Area} \approx 21.9776$

$$\Delta x = \frac{b-a}{n}$$

right sum

$n = 50$

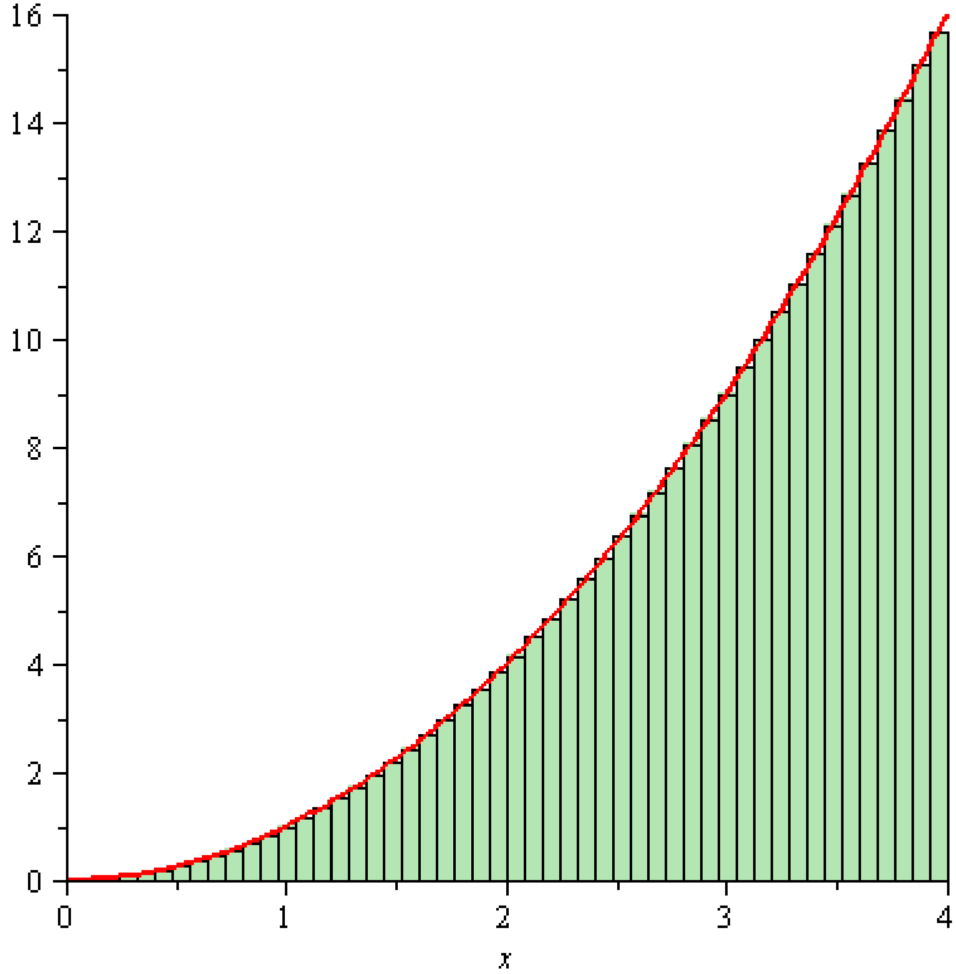


$f(x) = x^2$

And finally, here is the estimate for the middle riemann sum.

Area ≈ 21.3312

$$\Delta x = \frac{b-a}{n}$$



$f(x) = x^2$

middle sum

$n = 50$

As you can see, with 50 rectangles all of the estimates are close to 21, and the true area under the curve is 21 1/3.

$\text{Area} \approx 21.3312$

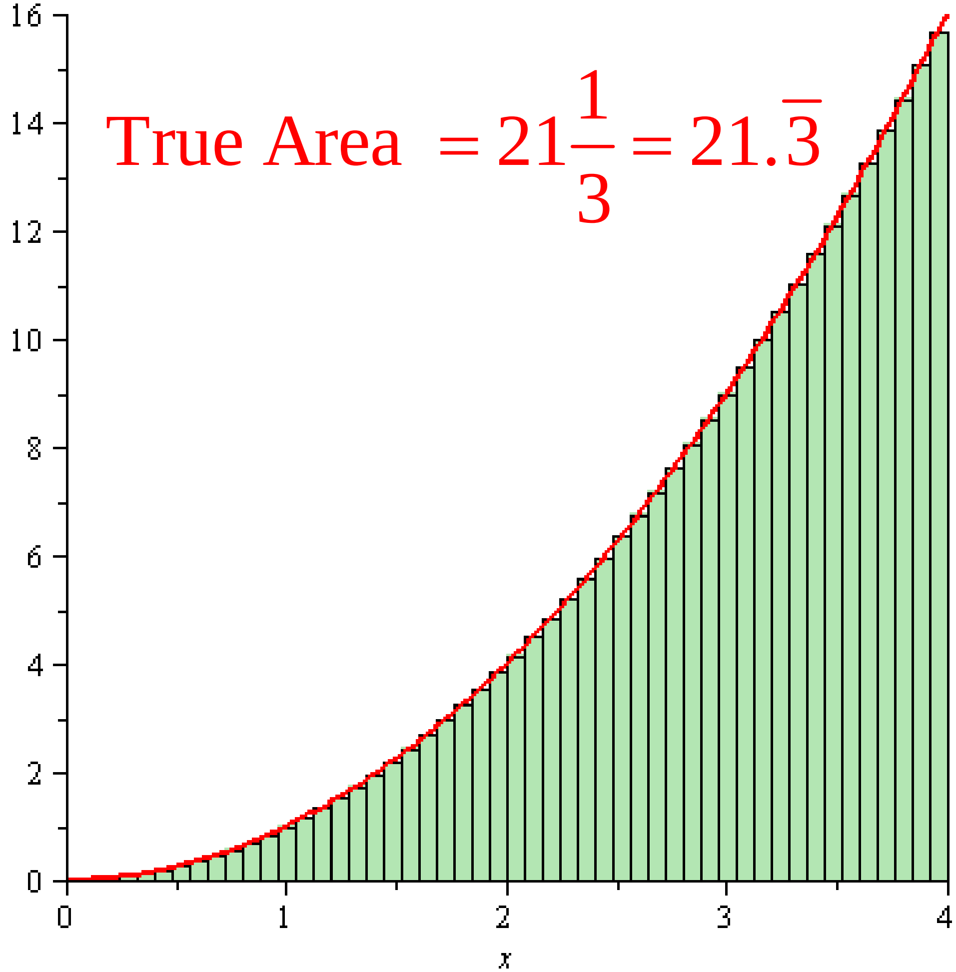
$$\Delta x = \frac{b-a}{n}$$

middle sum

$n = 50$

$\text{True Area} = 21\frac{1}{3} = 21.\overline{3}$

$f(x) = x^2$



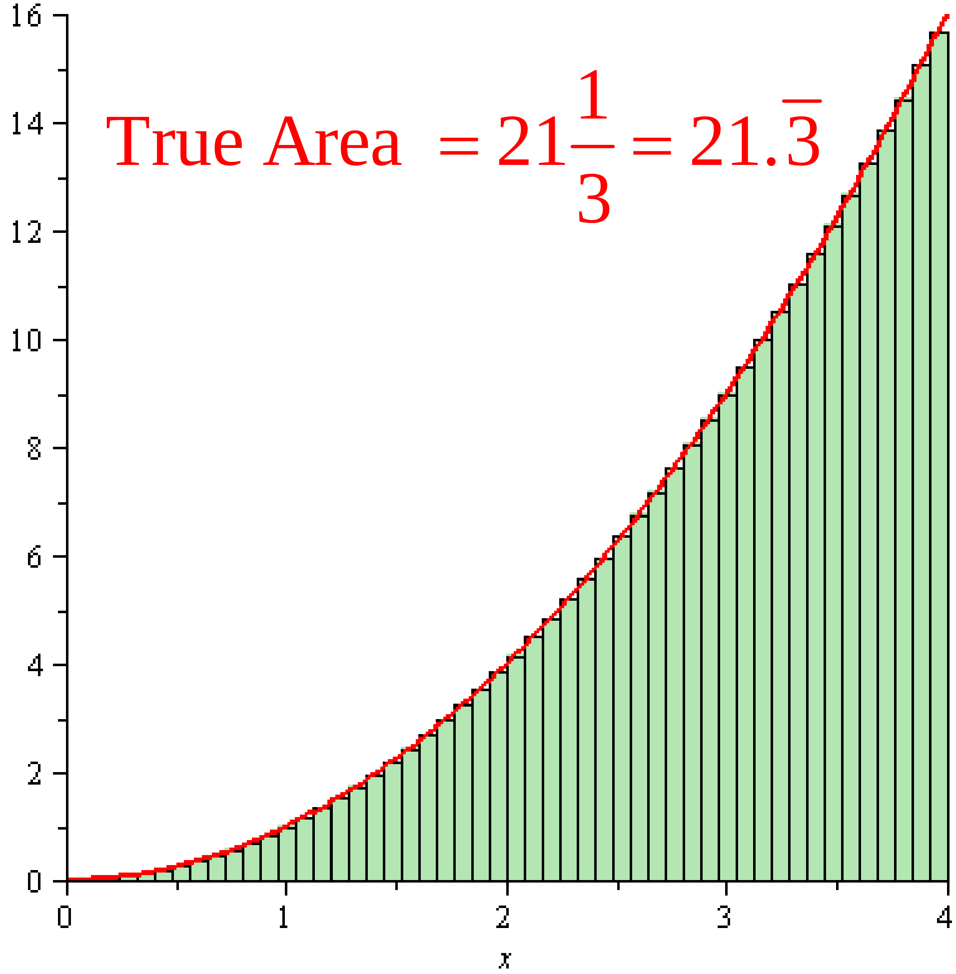
The true area under the curve is the result of a limit process as we let delta x go to zero.

$\text{Area} \approx 21.3312$

$\Delta x = \dfrac{b-a}{n}$

middle sum

$n = 50$



$\text{True Area } = 21\dfrac{1}{3} = 21.\overline{3}$

$f(x) = x^2$

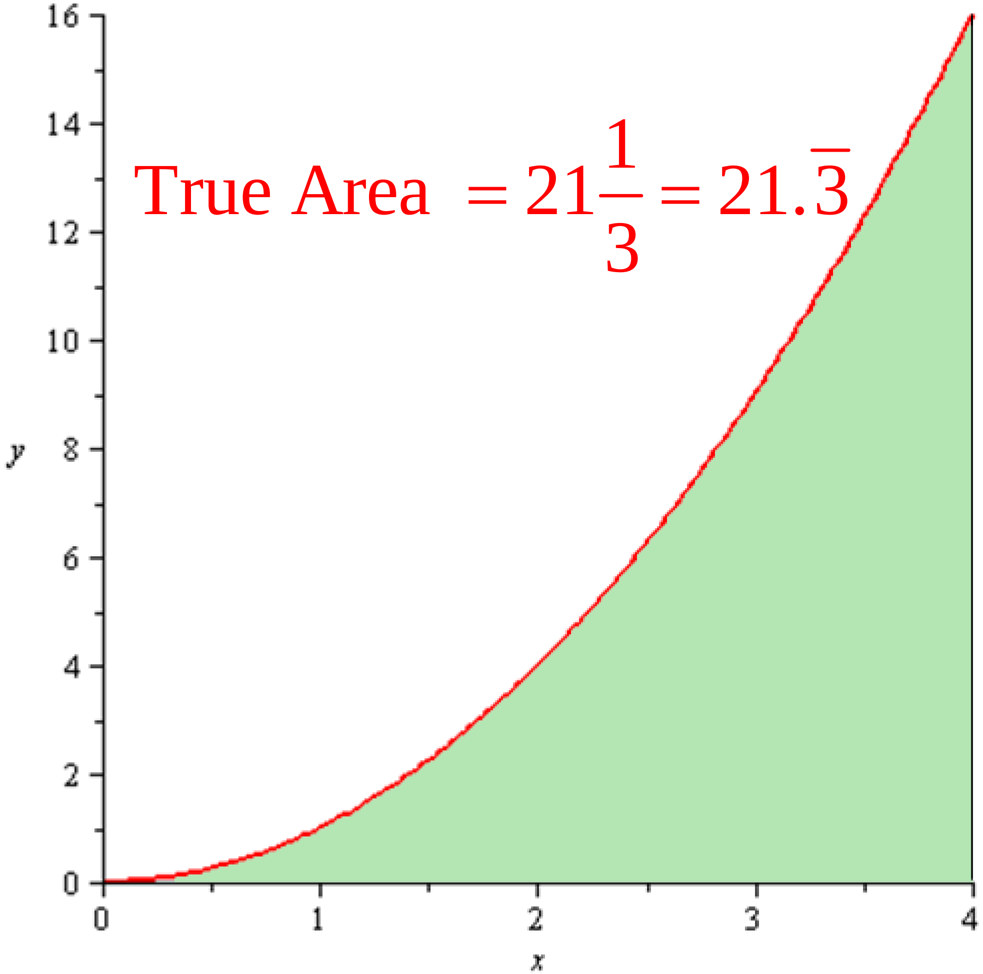
We call this limit the definite integral of our function as x goes from a to b.

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum f(x) \cdot \Delta x$$

$$\Delta x = \frac{b-a}{n}$$

$$f(x) = x^2$$

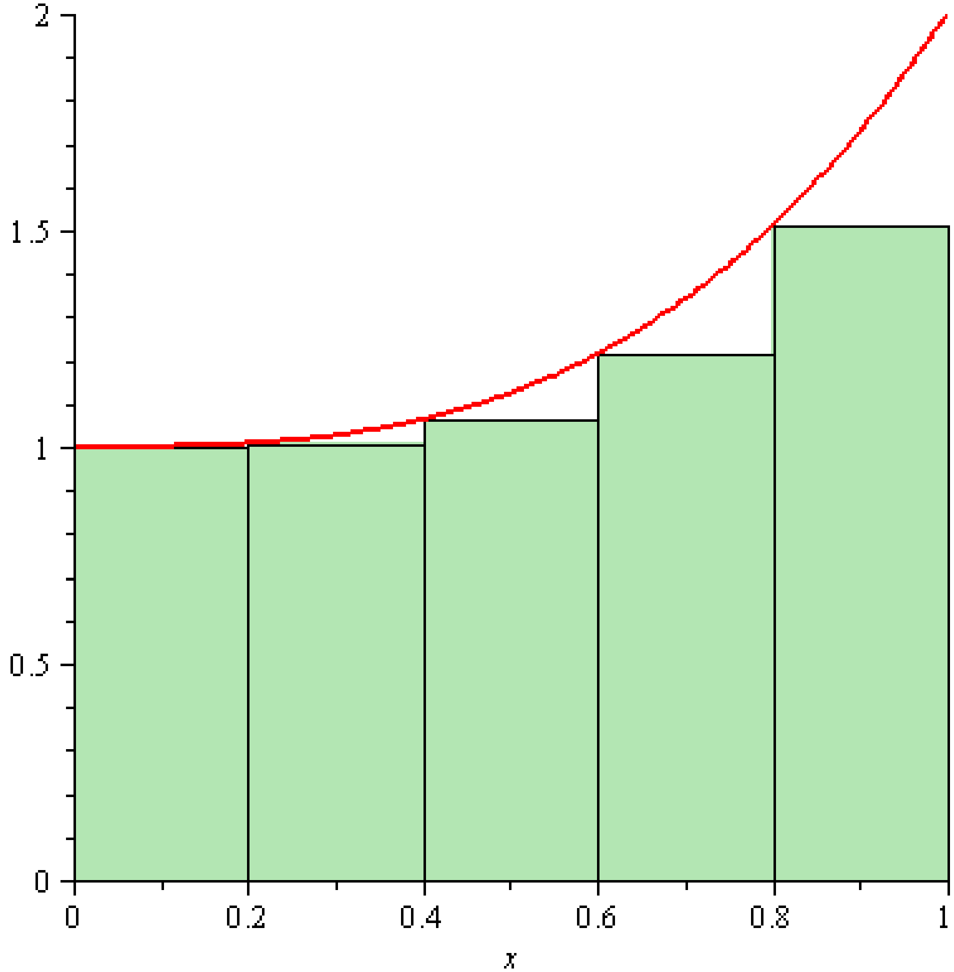
True Area $= 21\frac{1}{3} = 21.\overline{3}$



In this section, we'll always estimate the definite integral using a left riemann sum. Here's another example.

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{1-0}{5} = 0.2$$



$$f(x) = x^3 + 1$$

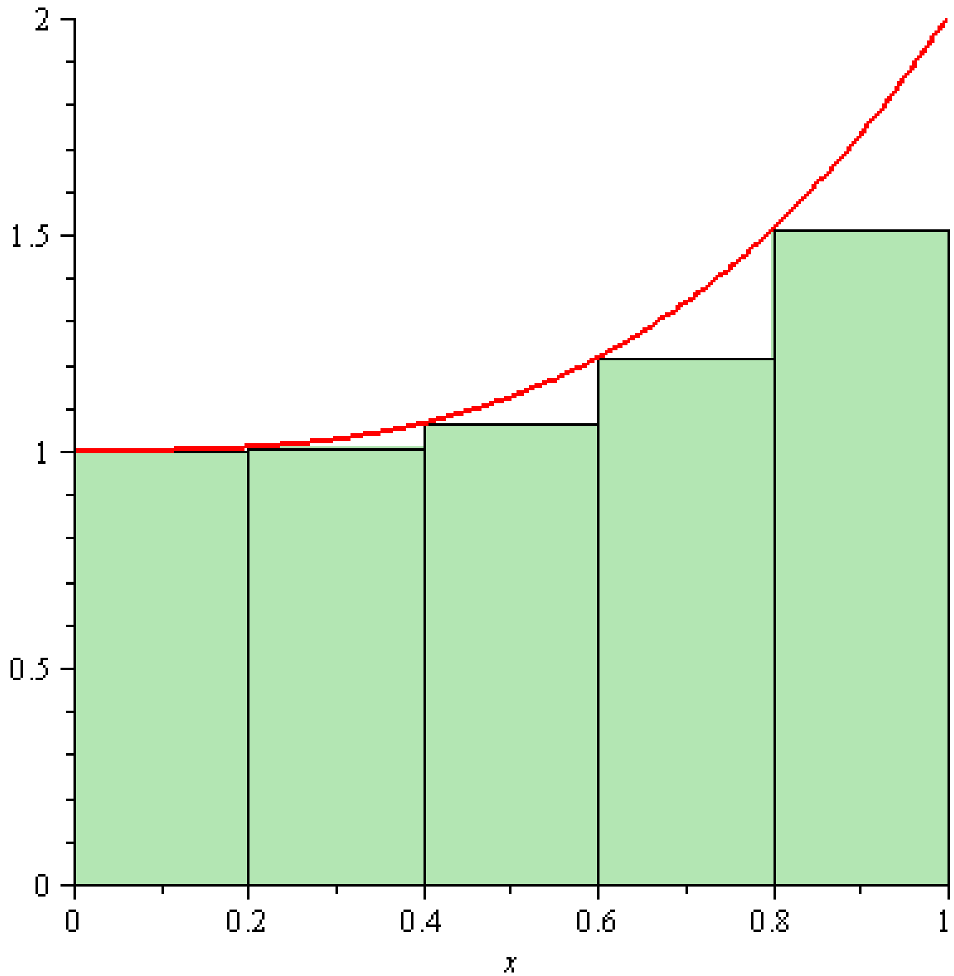
In this section, we'll always estimate the definite integral using a left riemann sum. Here's another example.

$$\int_0^1 (x^3+1)\,dx \approx f(0)\cdot .2 + f(.2)\cdot .2 + f(.4)\cdot .2 + f(.6)\cdot .2 + f(.8)\cdot .2$$

$$= 1.16$$

$$\Delta x = \frac{b-a}{n}$$

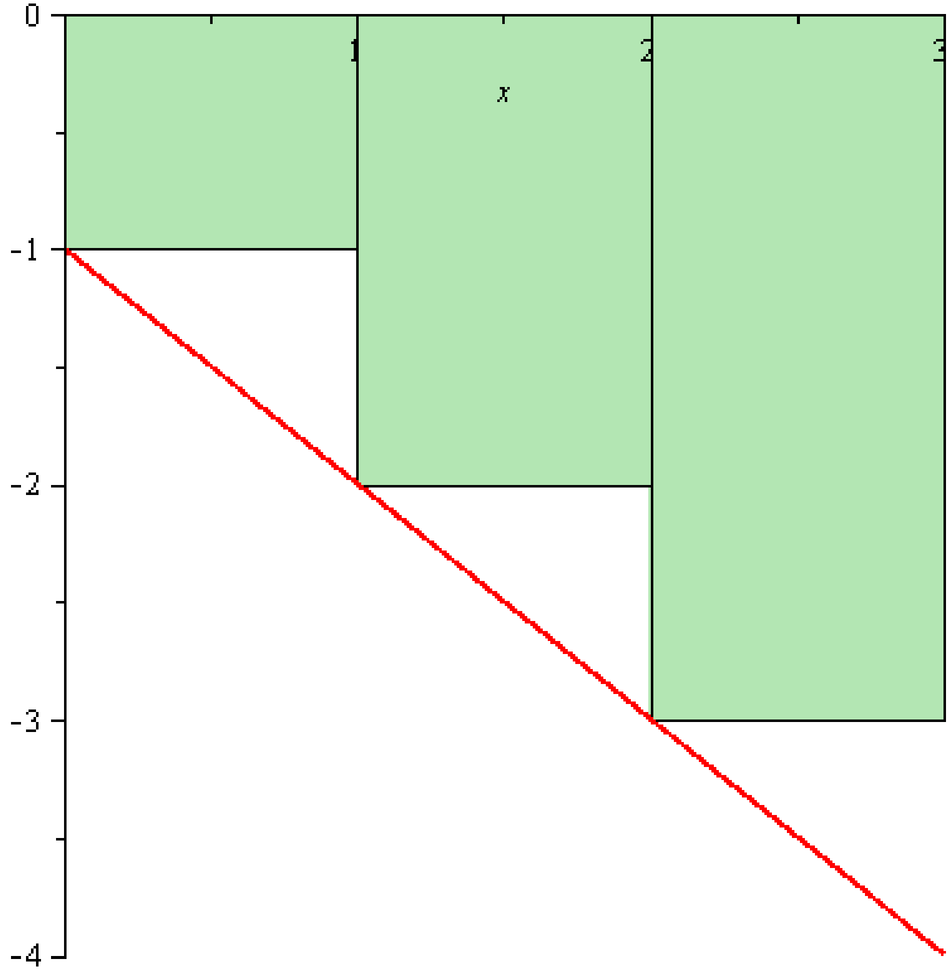
$$= \frac{1-0}{5} = 0.2$$



$$f(x) = x^3 + 1$$

Notice that if our curve is below the *x-axis*, then the value of the definite integral is negative.

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{3} = 1$$



$$f(x) = -x - 1$$

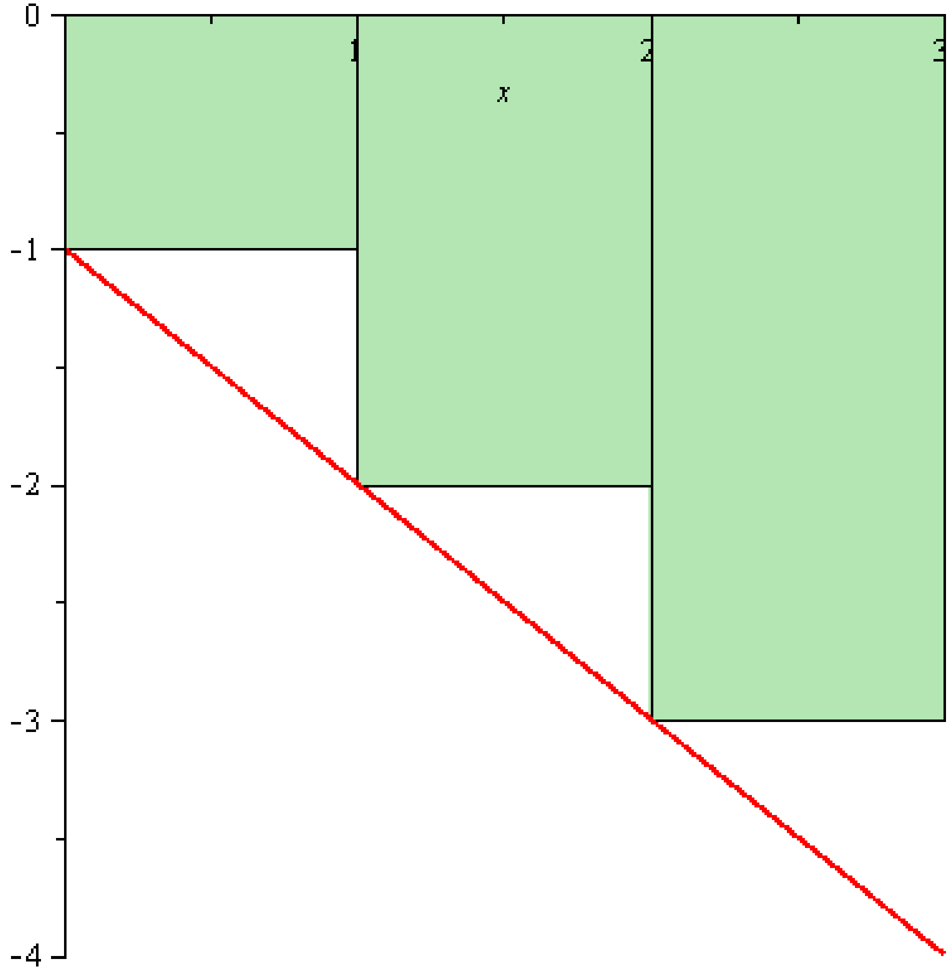
Notice that if our curve is below the *x-axis*, then the value of the definite integral is negative.

$$\int_0^3 (-x-1)\,dx \approx f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 = -1-2-3 = -6$$

$$f(x) = -x-1$$

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{3-0}{3} = 1$$



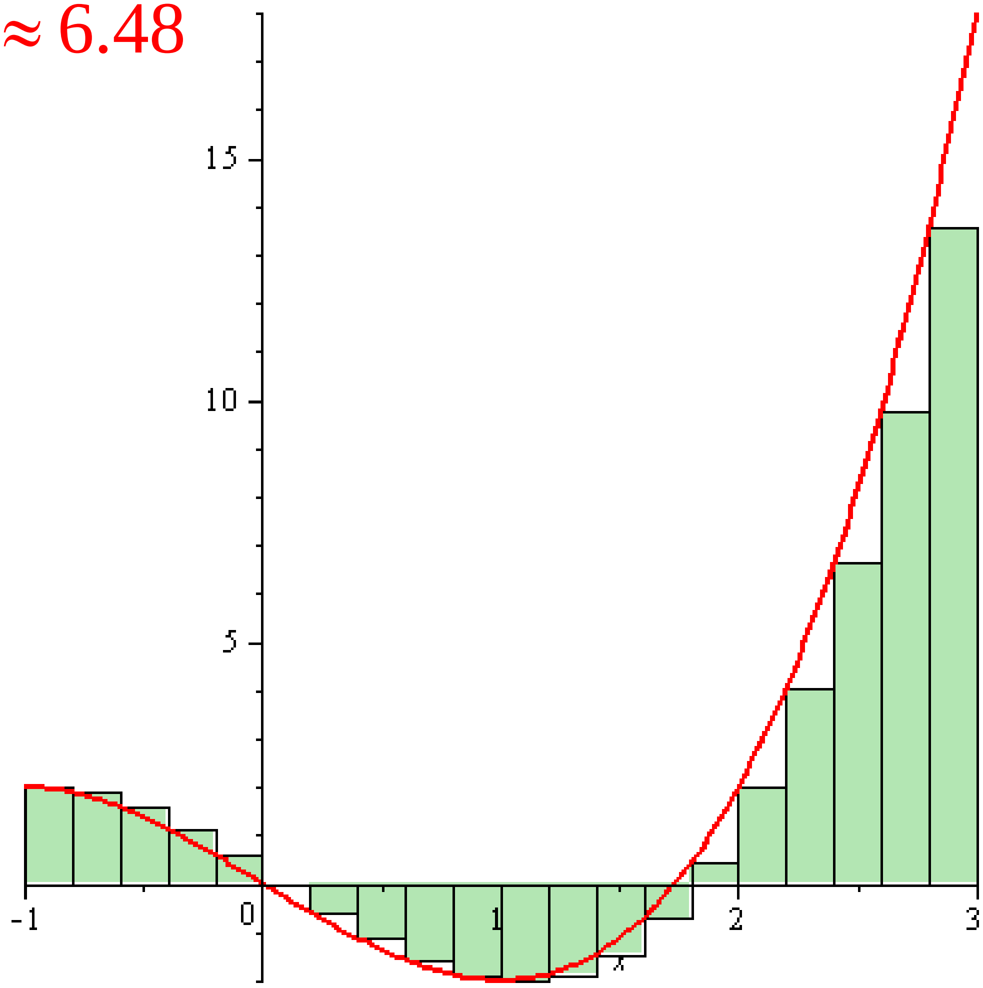
Also, if our function is partly above and partly below the *x-axis*, then the definite integral is a sum of both positive and negative areas.

$$\int_{-1}^{3} (x^3 - 3x)\, dx \approx 6.48$$

$$f(x) = x^3 - 3x$$

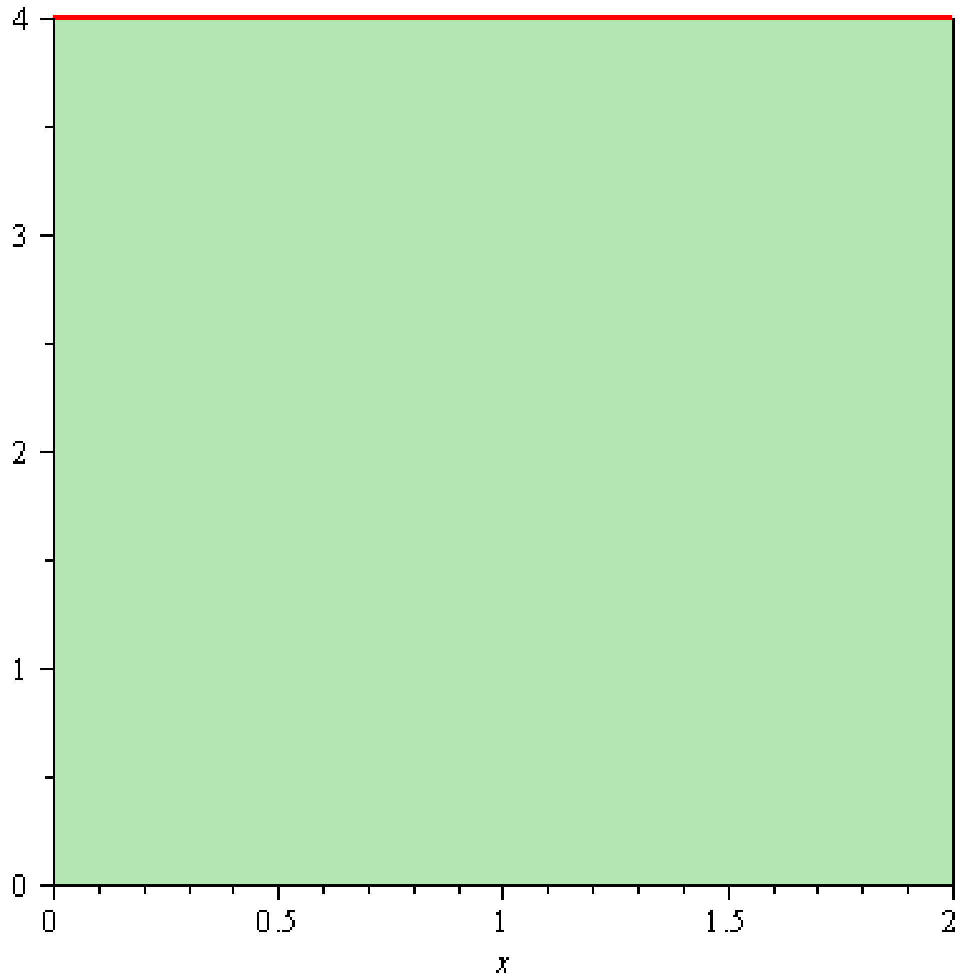
$$\Delta x = \frac{b-a}{n}$$

$$= \frac{3-(-1)}{20} = 0.$$



Now here's a practical application of the definite integral. Suppose you run at a constant 4 miles/hour for 2 hours. Then the distance you run is the same as the area under the curve below.

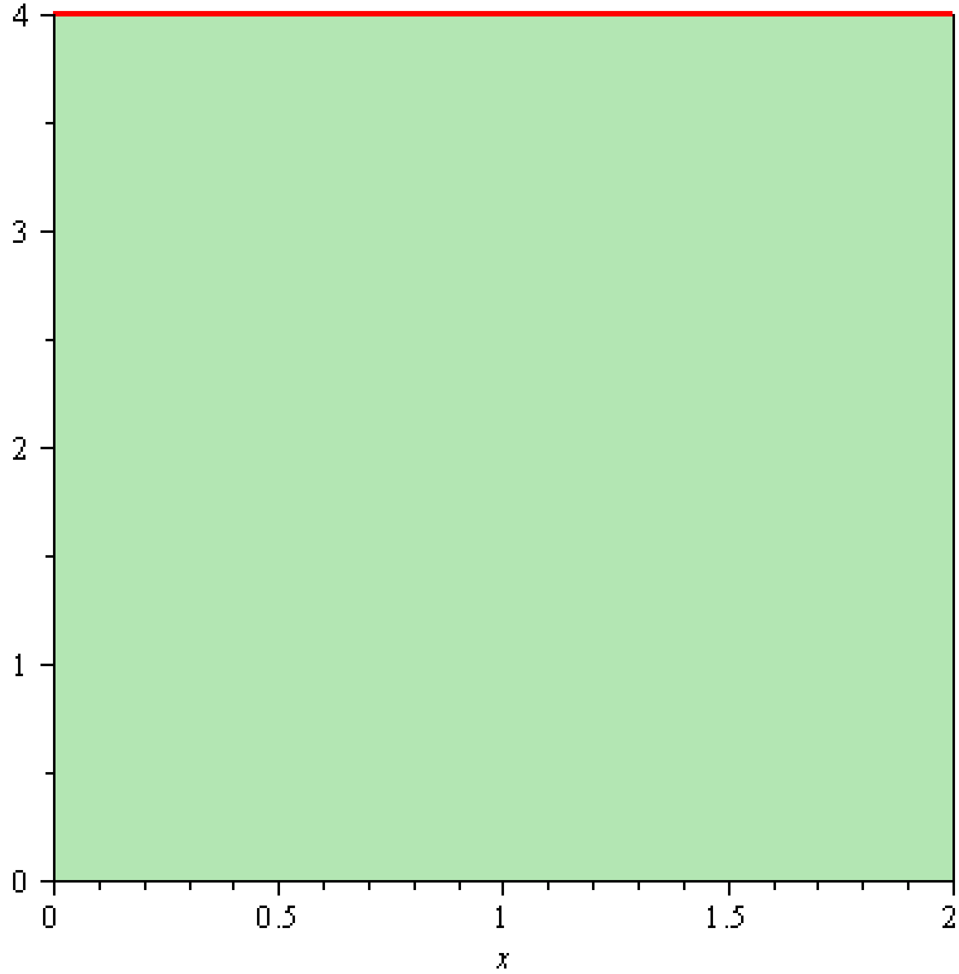
$$\int_0^2 4\,dx = 8$$



$$f(x) = 4$$

Notice that the units on our answer are the output units times the input units.

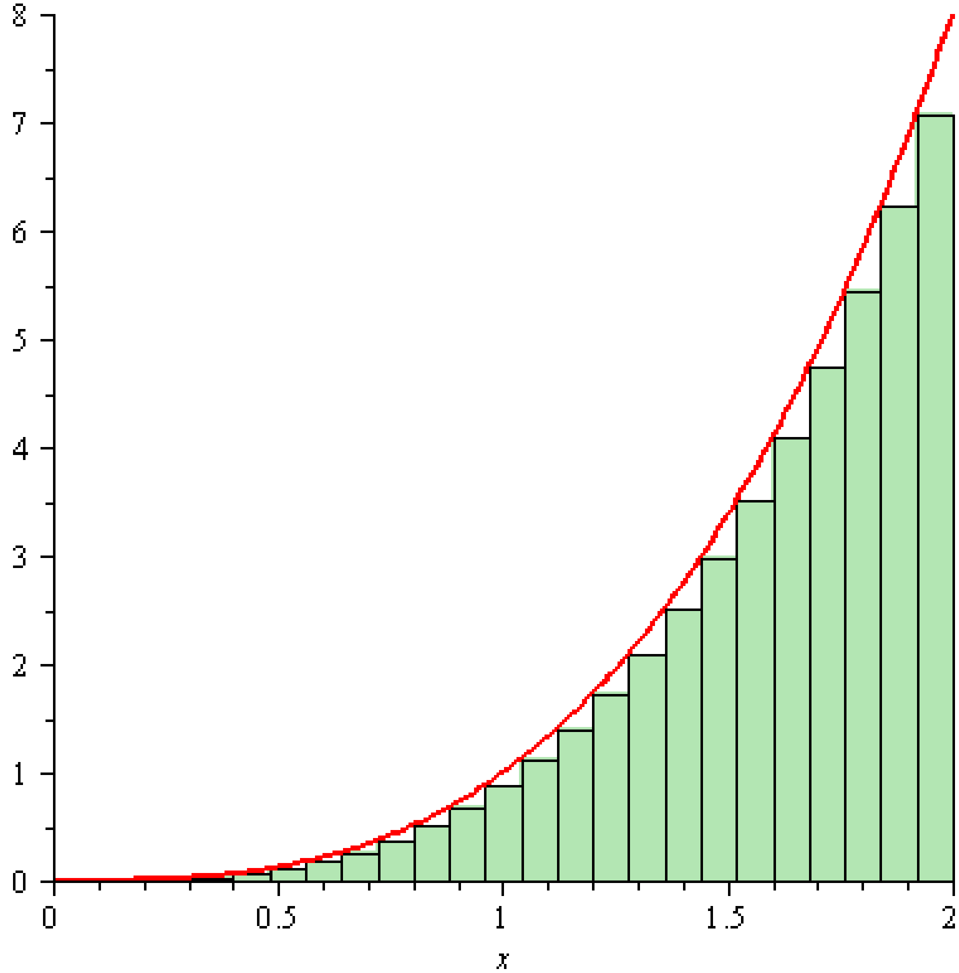
$$\int_0^2 4\frac{\text{miles}}{\text{hour}}\,dx\,\text{hours} = 8\,\text{miles}$$



$$f(x) = 4$$

Furthermore, if our speed is variable, then the total distance traveled is still equal to the area under the curve.

$$\int_0^2 x^3\,dx = \int_0^2 x^3 \frac{\text{miles}}{\text{hour}}\,dx\,\text{hours} = 4\,\text{miles}$$



$$f(x) = x^3$$