INTEGRATION BY SUBSTITUTON

$$du = \frac{du}{dx}dx$$

Once one has learned the basic rules for finding derivatives, the process is pretty simple.

1. $\frac{d(c)}{dx}=0$

2. $\frac{dx^n}{dx}=nx^{n-1}$

3. $\frac{d(cf(x))}{dx}=c\frac{df}{dx}$

4. $\frac{d(f\pm g)}{dx}=\frac{df}{dx}\pm\frac{dg}{dx}$

5. $\frac{df(g(x))}{dx}=\frac{df}{dg}\frac{dg}{dx}$

6. $\frac{d(fg)}{dx}=f\frac{dg}{dx}+g\frac{df}{dx}$

7. $\frac{d\left(\frac{f}{g}\right)}{dx}=\frac{g\frac{df}{dx}-f\frac{dg}{dx}}{g^2}$

8. $\frac{de^x}{dx}=e^x,\ \frac{db^x}{dx}=b^x\ln b$

9. $\frac{d\ln x}{dx}=\frac{1}{x},\ \frac{d\log_b x}{dx}=\frac{1}{x\ln b},\ x>0$

10. $\frac{d\ln|x|}{dx}=\frac{1}{x},\ x\neq 0$

Unfortunately, the process for finding antiderivatives is not always quite so straight forward.

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$

2. $\int x^{-1} \, dx = \ln|x| + c$

3. $\int e^x \, dx = e^x + c$

4. $\int b^x \, dx = \frac{b^x}{\ln b} + c$

5. $\int kf(x) \, dx = k \int f(x) \, dx \quad (k \text{ a constant})$

6. $\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

For example, while the integral or antiderivative of the following function is easy to do, …

$$\int x^{99}\, dx = \frac{x^{100}}{100} + c$$

For example, while the integral or antiderivative of the following function below is easy to find, …

$$\int x^{99}\,dx = \frac{x^{100}}{100} + c$$

This next example, on the other hand, is not so simple.

$$\int x\left(x^2+1\right)^{99}\,dx = ?$$

Fortunately, a technique call *integration by substitution* can help us solve many problems of this sort.

$$\int x\left(x^2+1\right)^{99}dx=?$$

Since our function looks somewhat like *x^{99}*, we're going to try to make it look exactly like that by substituting *u* for *$x^2 + 1$*.

$$\int x\left(x^2+1\right)^{99} dx = ?$$

$$u = x^2 + 1$$

However, if we start expressing our function in terms of *u*, then we are also going to have to express *dx* in terms of *u*. We begin this process by first finding *du/dx*.

$$\int x\left(x^2+1\right)^{99}dx=?$$

$$u=x^2+1$$

$$\frac{du}{dx}=2x$$

Next, we treat *du/dx* as a fraction and algebraically solve for *dx* in terms of *du*. Also, we'll often call these symbols *differential x* and *differential u*.

$$\int x\left(x^2+1\right)^{99} dx = ?$$

$$u = x^2 + 1$$

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{1}{2x}du$$

We now make our substitutions into the original integral, and if this process is going to work, then any remaining occurrences of *x* will cancel out.

$$\int x\left(x^2+1\right)^{99}dx=\int x\cdot u^{99}\cdot\frac{1}{2x}du=\frac{1}{2}\int u^{99}\,du$$

$$u=x^2+1$$

$$\frac{du}{dx}=2x\Rightarrow du=2x\,dx\Rightarrow dx=\frac{1}{2x}du$$

This last integral is now easy to do. However, the final step is to, after integrating, put everything back in terms of *x*.

$$\int x\left(x^2+1\right)^{99} dx = \int x \cdot u^{99} \cdot \frac{1}{2x} du = \frac{1}{2}\int u^{99}\, du$$

$$u = x^2 + 1$$

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{1}{2x} du$$

And it's just that easy!

$$\int x\left(x^2+1\right)^{99} dx = \int x \cdot u^{99} \cdot \frac{1}{2x} du = \frac{1}{2}\int u^{99}\, du$$

$$= \frac{1}{2} \cdot \frac{u^{100}}{100} + c = \frac{\left(x^2+1\right)^{100}}{200} + c$$

$$u = x^2 + 1$$

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{1}{2x} du$$

To check our answer, we can simply differentiate our result and see if we get back the original integrand.

$$\int x\left(x^2+1\right)^{99}dx=\int x\cdot u^{99}\cdot\frac{1}{2x}du=\frac{1}{2}\int u^{99}\,du$$

$$=\frac{1}{2}\cdot\frac{u^{100}}{100}+c=\frac{\left(x^2+1\right)^{100}}{200}+c$$

$$\frac{d\,\dfrac{\left(x^2+1\right)^{100}}{200}}{dx}=\frac{1}{200}\cdot 100\left(x^2+1\right)^{99}\cdot 2x=x\left(x^2+1\right)^{99}$$

The trick to *integration by substituion* is finding just the right substitution, and keep in mind that you are trying to change your integral to something simpler that you already know how to integrate. Here are a few more examples.

$$\int 4x\left(x^2+1\right)^6 dx$$

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$$u = x^2 + 1$$

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{1}{2x}du$$

$$\int 4x\left(x^2+1\right)^6 dx = \int 4x \cdot u^6 \cdot \frac{1}{2x} du = 2\int u^6 du$$

$$= 2 \cdot \frac{u^7}{7} + c = \frac{2(x^2+1)^7}{7} + c$$

$$u = x^2 + 1$$

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{1}{2x} du$$

$$\int x^2\left(x^3+1\right)^2 dx$$

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$$\frac{du}{dx} = 3x^2 \Rightarrow du = 3x^2\,dx \Rightarrow dx = \frac{1}{3x^2}du$$

$$\int x^2\left(x^3+1\right)^2 dx=\int x^2\cdot u^2\cdot\frac{1}{3x^2}du=\frac{1}{3}\int u^2\,du$$

$$=\frac{1}{3}\cdot\frac{u^3}{3}+c=\frac{(x^3+1)^3}{9}+c$$

$$u=x^3+1$$

$$\frac{du}{dx}=3x^2\Rightarrow du=3x^2\,dx\Rightarrow dx=\frac{1}{3x^2}du$$

$$\int 3xe^{x^2}\,dx$$

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$$u = x^2$$

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$$u = x^2$$

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{1}{2x}du$$

$$\int 3xe^{x^2}\,dx = \int 3xe^{u}\cdot\frac{1}{2x}du = \frac{3}{2}\int e^{u}\,du = \frac{3}{2}e^{u} + c$$

$$= \frac{3}{2}e^{x^2} + c$$

$$u = x^2$$

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{1}{2x}du$$

$$\int \frac{1}{2x+5}\,dx$$

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$$u = 2x+5$$

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$$\frac{du}{dx} = 2 \Rightarrow du = 2\,dx \Rightarrow dx = \frac{1}{2}du$$

$$\int \frac{1}{2x+5}\,dx = \int \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|2x+5| + c$$

$$u = 2x+5$$

$$\frac{du}{dx} = 2 \Rightarrow du = 2\,dx \Rightarrow dx = \frac{1}{2}\,du$$

$$\int (x+3)\sqrt{x^2+6x}\,dx$$

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$$u = x^2 + 6x$$

$$\int (x+3)\sqrt{x^2+6x}\,dx$$

$$u = x^2 + 6x$$

$$\frac{du}{dx} = 2x+6 \Rightarrow du = (2x+6)\,dx \Rightarrow dx = \frac{1}{2x+6}du$$

$$\int (x+3)\sqrt{x^2+6x}\,dx = \int (x+3)\sqrt{u}\cdot\frac{1}{2x+6}\,du$$

$$= \int (x+3)\cdot u^{1/2}\cdot\frac{1}{2(x+3)}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{u^{3/2}}{3/2}+c$$

$$= \frac{1}{2}\cdot\frac{2}{3}(x^2+6x)^{3/2}+c = \frac{1}{3}(x^2+6x)^{3/2}+c$$

$$u = x^2+6x$$

$$\frac{du}{dx} = 2x+6 \Rightarrow du = (2x+6)\,dx \Rightarrow dx = \frac{1}{2x+6}\,du$$

$$\int \frac{2x}{(x-5)^2}\,dx$$

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uh-oh!

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uh-oh!

$$u = x - 5 \Rightarrow x = u + 5$$

$$\frac{du}{dx} = 1 \Rightarrow du = dx$$

$$\int \frac{2x}{(x-5)^2}\,dx = \int \frac{2x}{u^2}\,du = \int 2x \cdot u^{-2}\,du$$

$$= \int 2(u+5)u^{-2}\,du = \int (2u^{-1} + 10u^{-2})\,du$$

$$= \int \left(\frac{2}{u} + 10u^{-2} \right) du = 2\ln|u| - \frac{10}{u} + c$$

$$= 2\ln|x-5| - \frac{10}{x-5} + c$$

$$u = x - 5 \Rightarrow x = u + 5$$

$$\frac{du}{dx} = 1 \Rightarrow du = dx$$

And now for a word problem from the book!