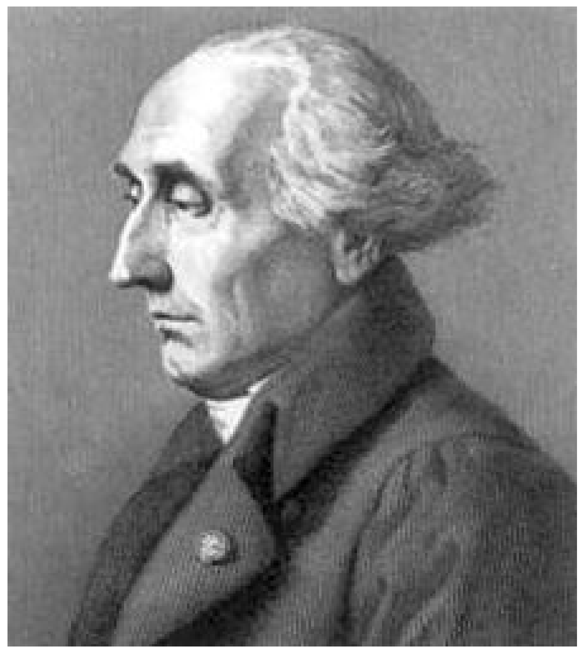
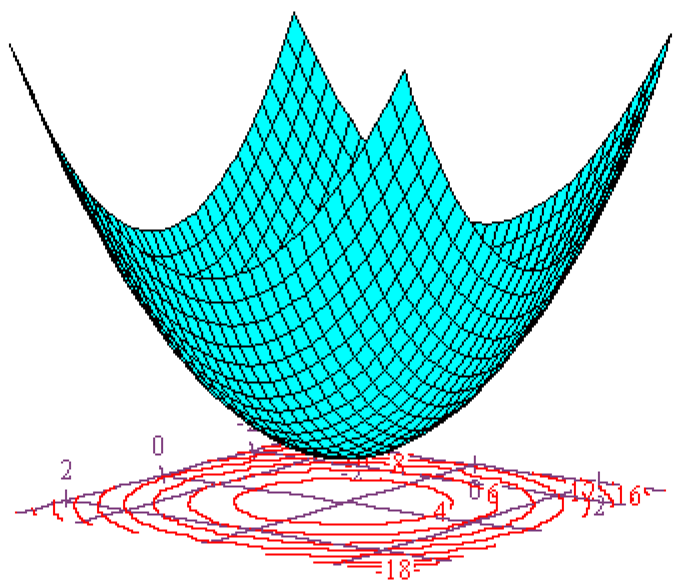
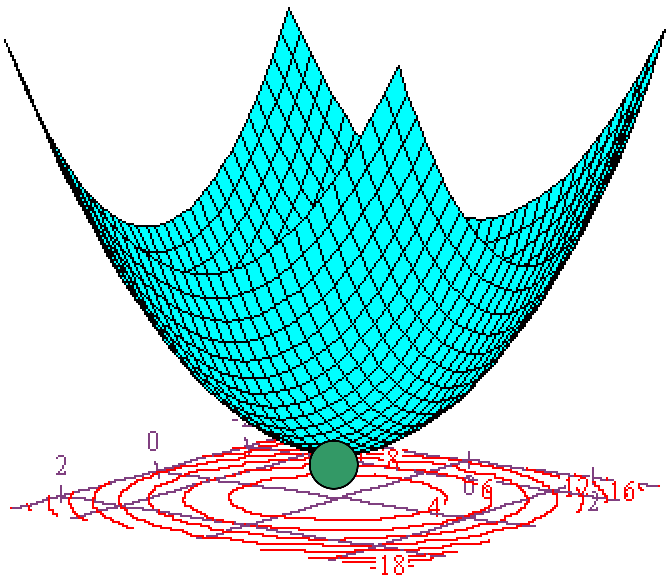
LAGRANGE MULTIPLIERS



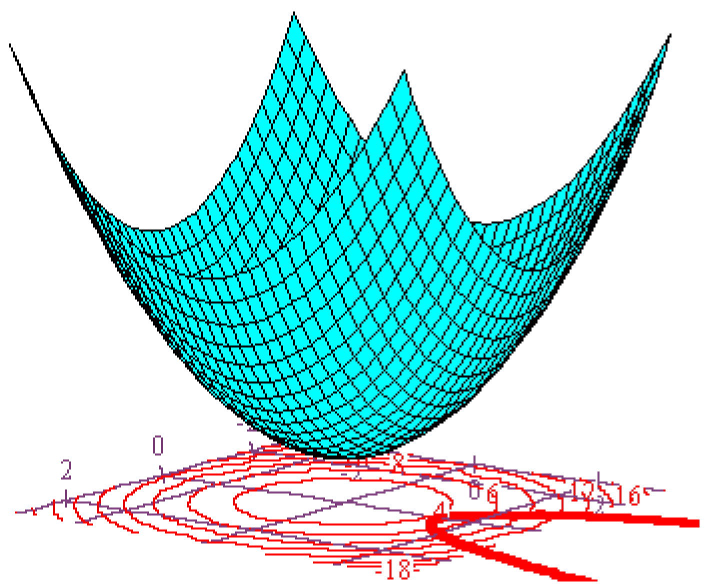
Let's start with a simple surface, *z=f(x,y)*.



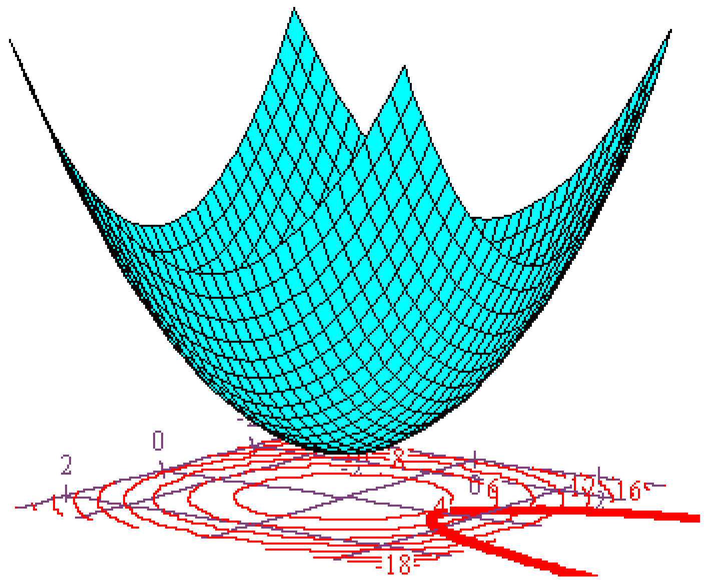
Clearly, this surface has a minimum point.



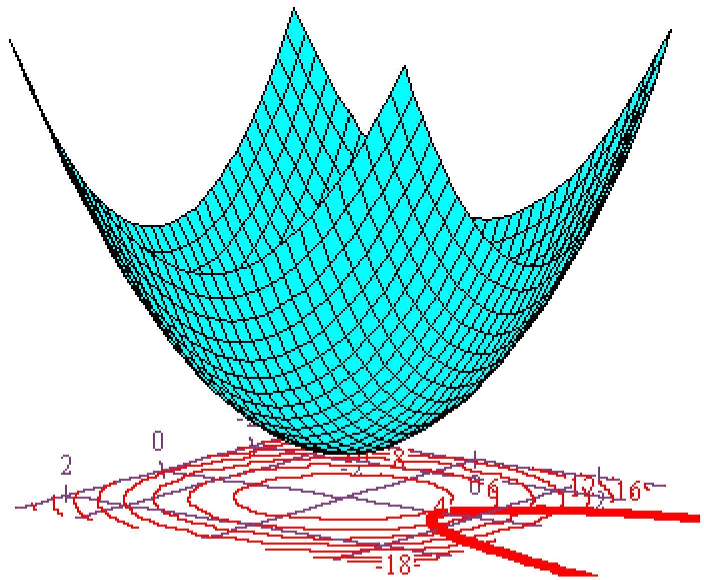
Now, down in the *xy*-plane, let's add a curve, *g(x,y)=c*.



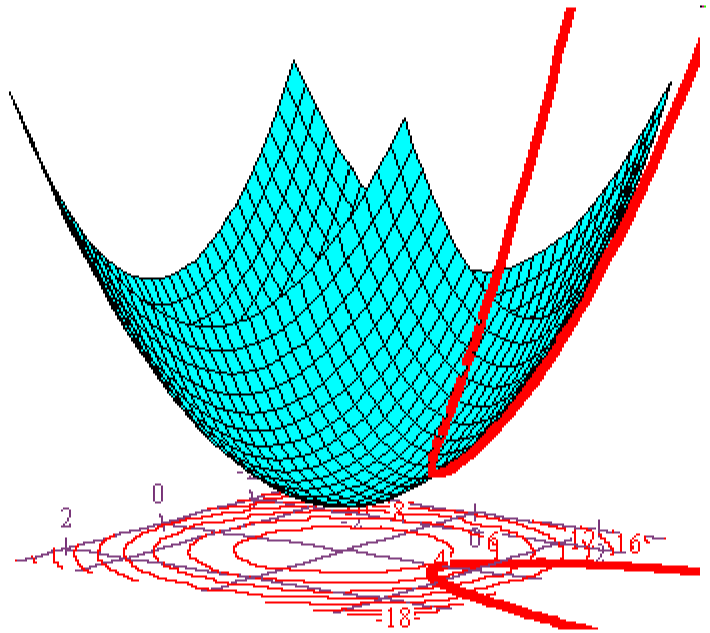
We can think of this curve as a level curve for a more general surface graph, *g=g(x,y)*.



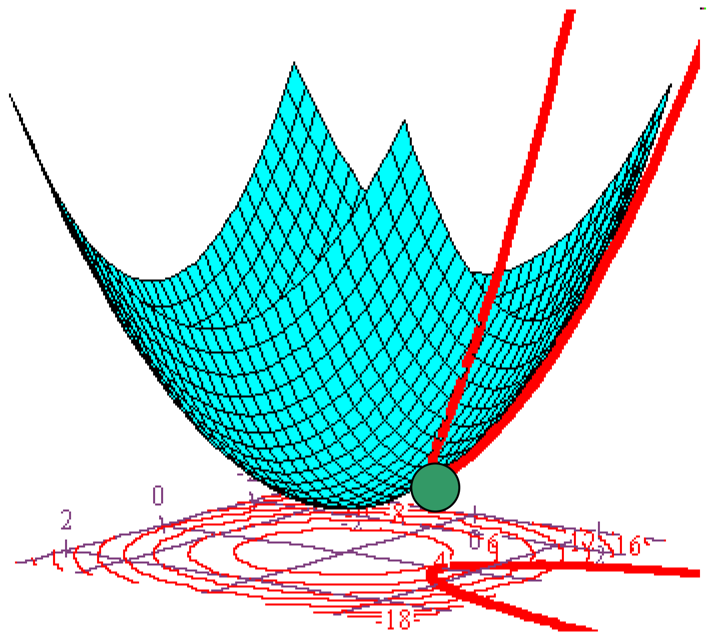
We can also think of this curve as representing a constraint on the values for *x* and *y* that we can plug into our function *z=f(x,y)*.



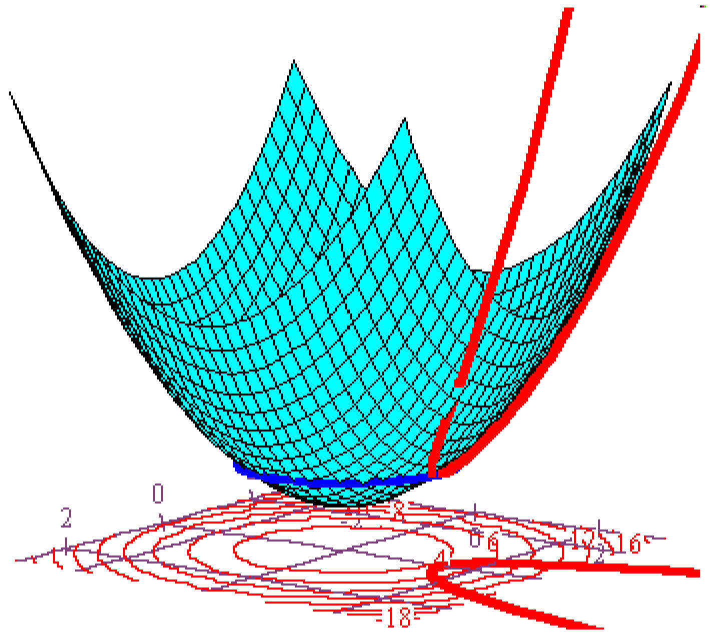
If we restrict the domain of *z=f(x,y)* to the curve *g(x,y)=c*, then the graph that results is just a curve lying on our original surface.



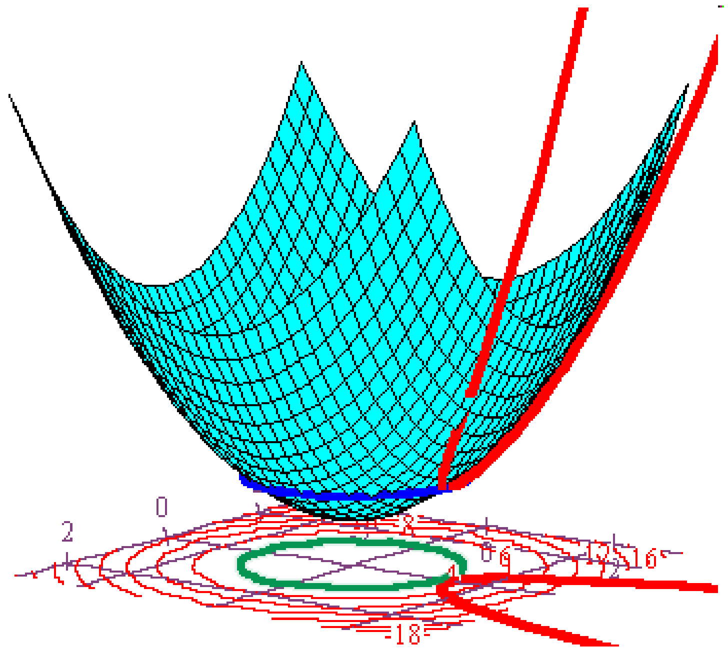
In this particular case, it’s easy to see that this curve has its own minimum point.

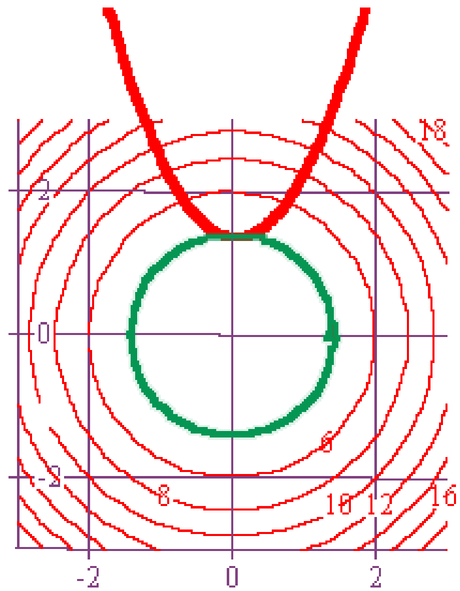


It's also easy to see that there is a contour, *z=k*, that touches our curve at that minimum point.

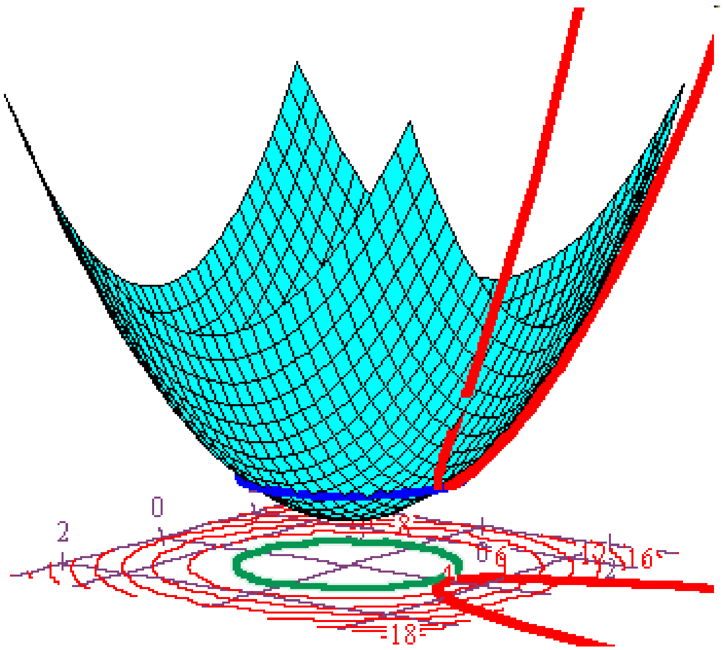


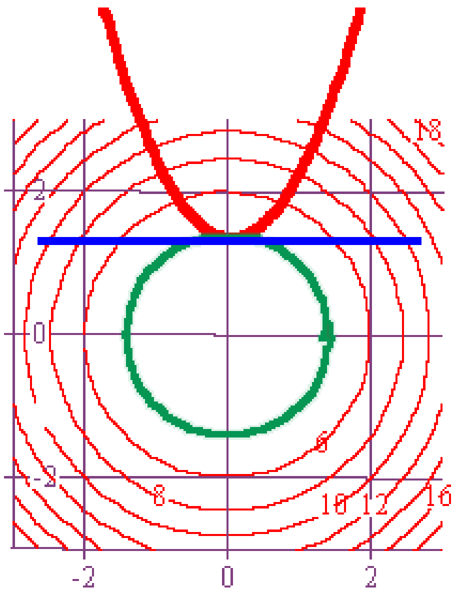
If we look at the level curve for this contour, we see that it is tangent to the curve *g(x,y)=c* in the *xy*-plane.



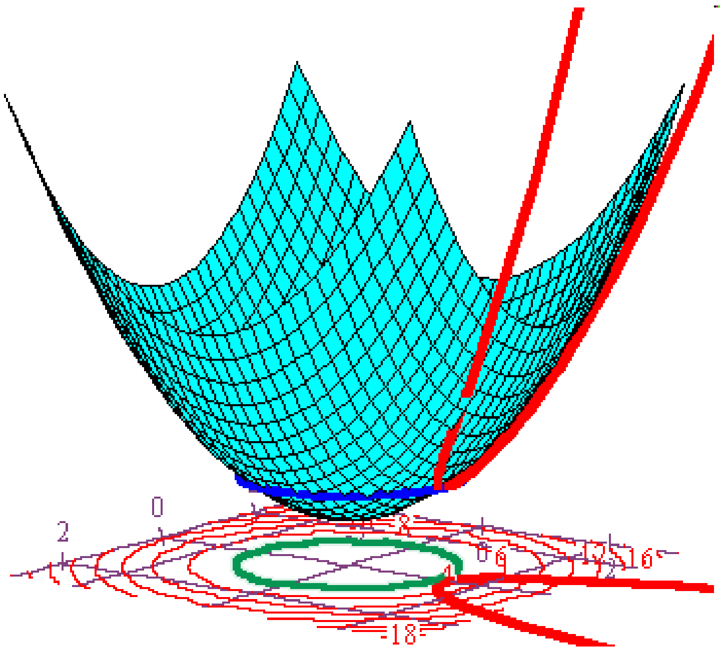


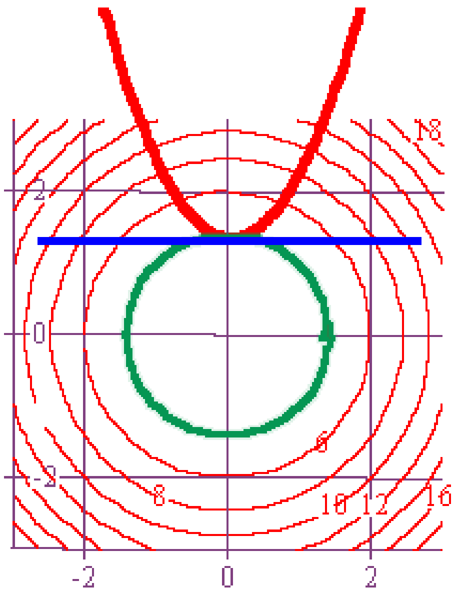
Hence, our level curve and *g(x,y)=c* have a common tangent line in the *xy*-plane.



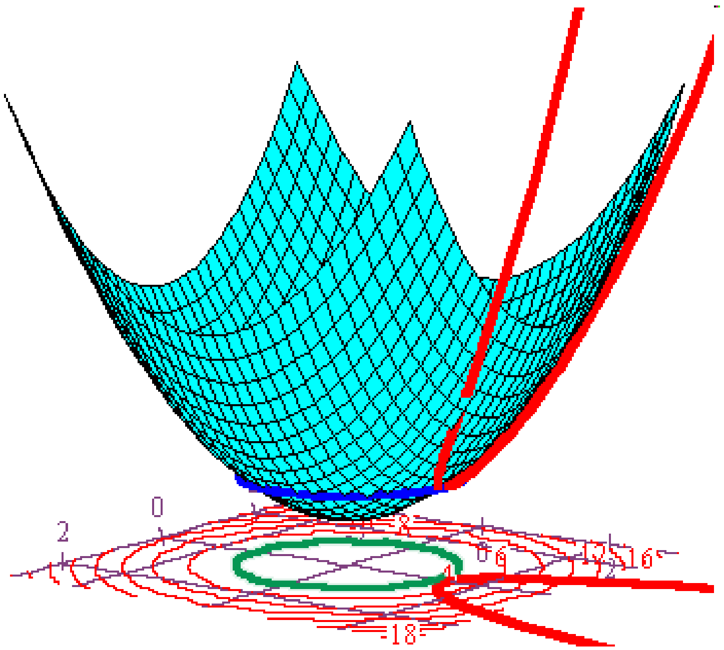


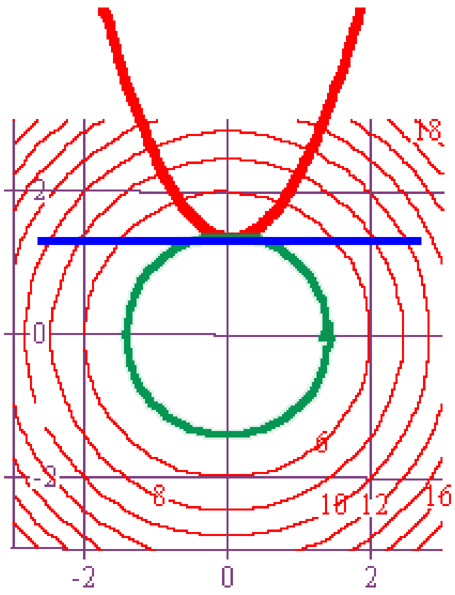
Let’s think about what this means.



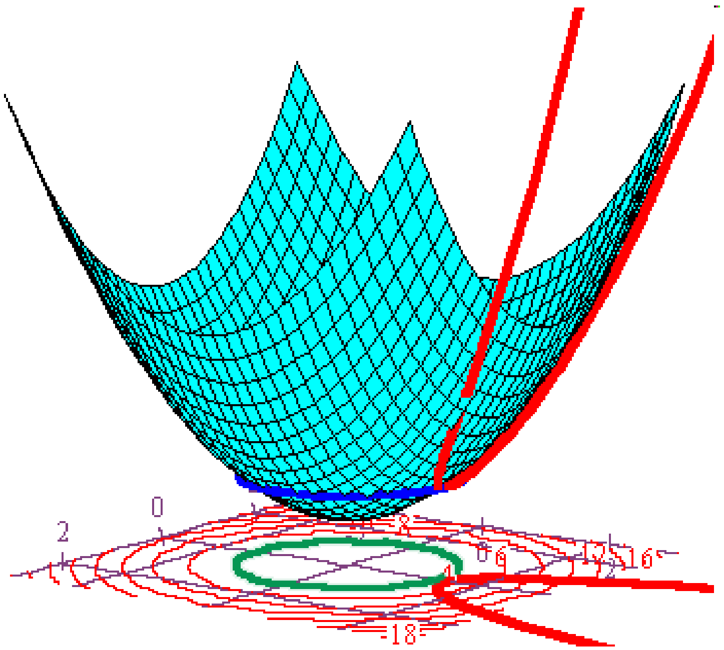


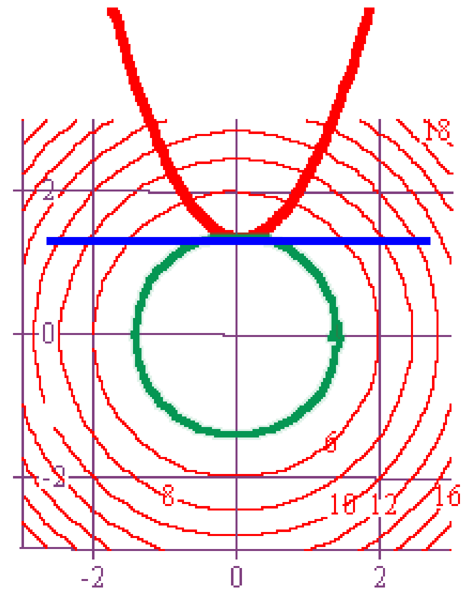
We know that derivatives have something to do with tangent lines.



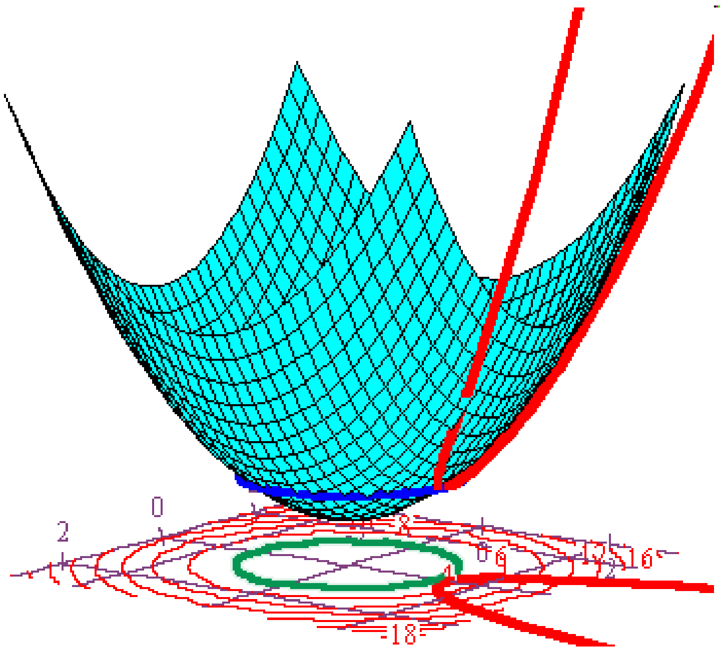


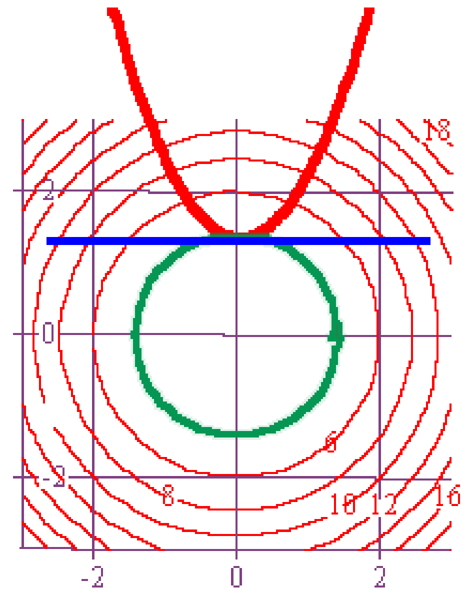
Hence, since our level curves have a common tangent line, we suspect there is a relationship between the partial derivatives of *z=f(x,y)* and *g=gx,y)*.



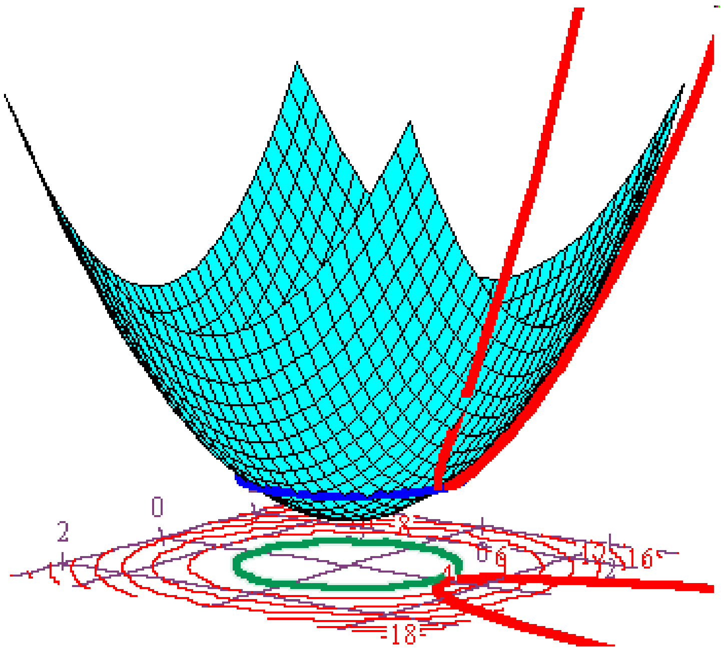


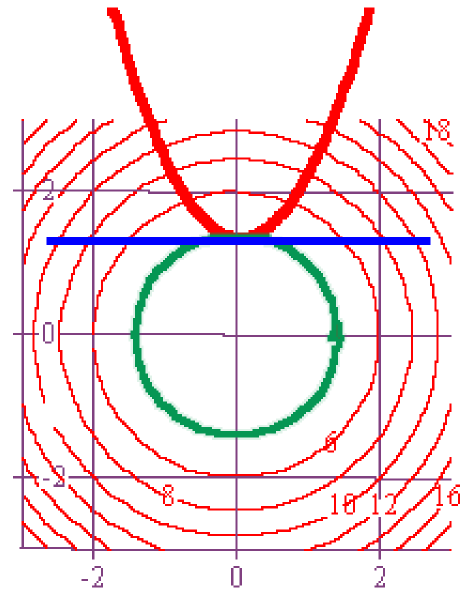
There is indeed a relationship!



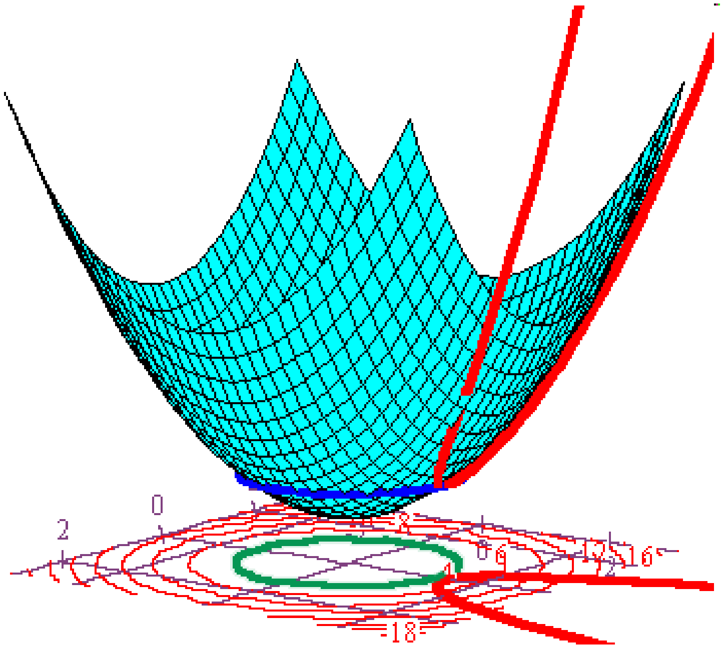


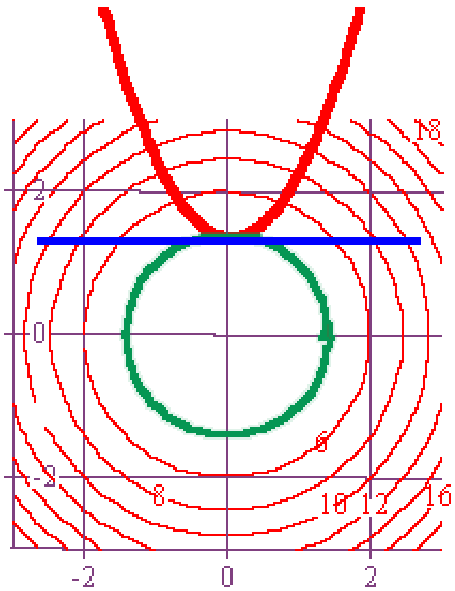
According to a theorem of Lagrange, at the minimum point of the particular curve on our surface, the partial derivatives of $z=f(x,y)$ will be a fixed multiple of the partial derivatives of $g=g(x,y)$.





We denote this multiplier by the Greek letter *lambda*, and the result is the following set of equations to solve.





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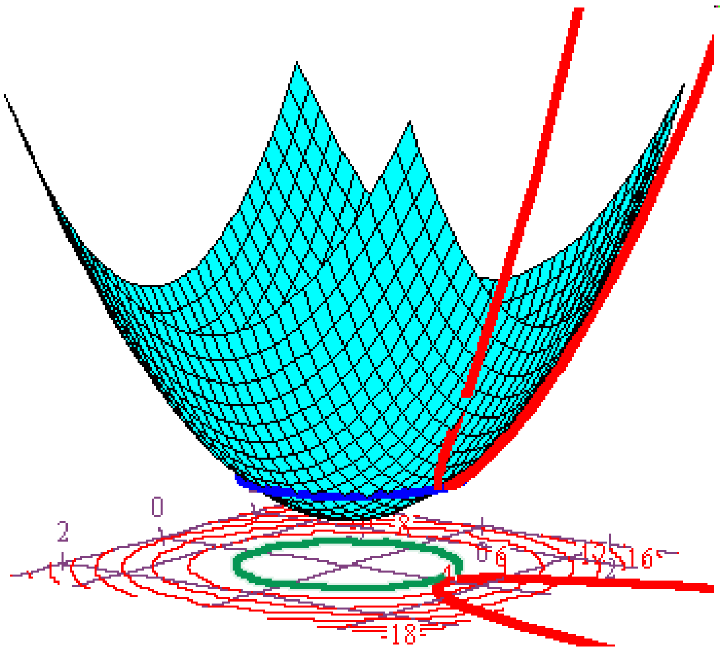
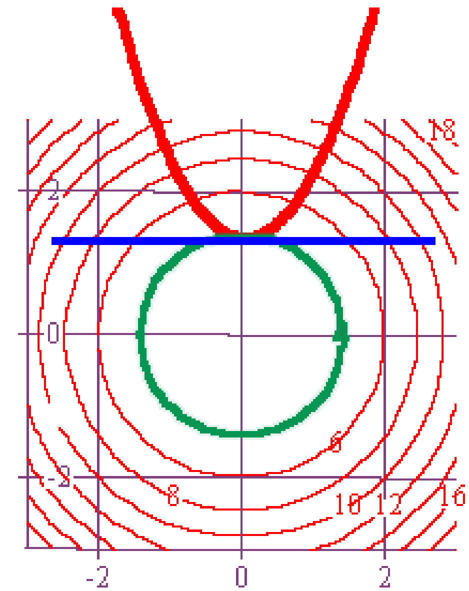


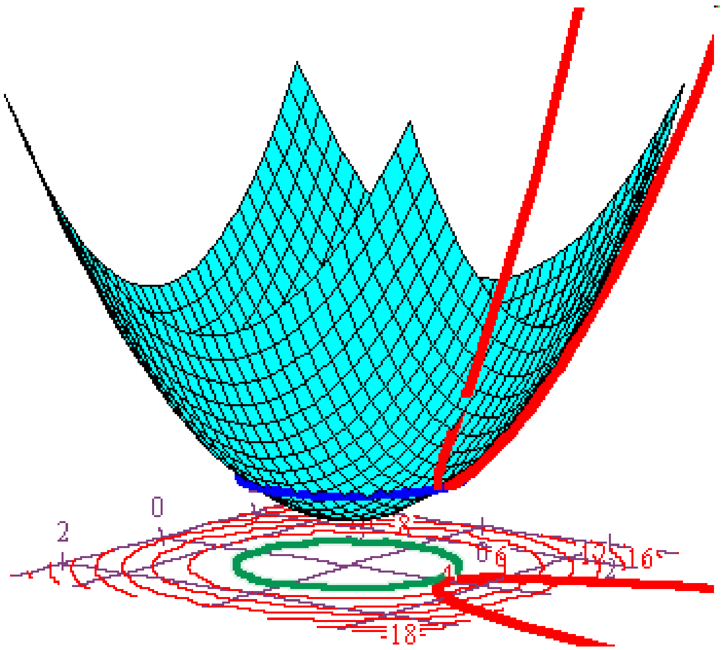


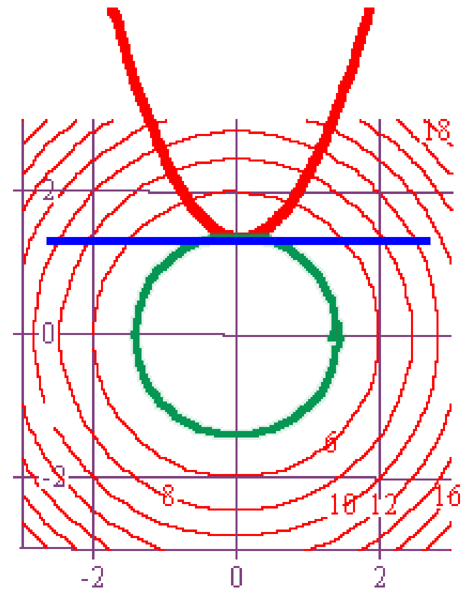
$$z_x = \lambda \cdot g_x$$

$$z_y = \lambda \cdot g_y$$

$$g(x, y) = c$$

We denote this multiplier by the Greek letter *lambda*, and the result is the following set of equations to solve.





Solve this system of equations and you're done.

$$z_x = \lambda \cdot g_x$$

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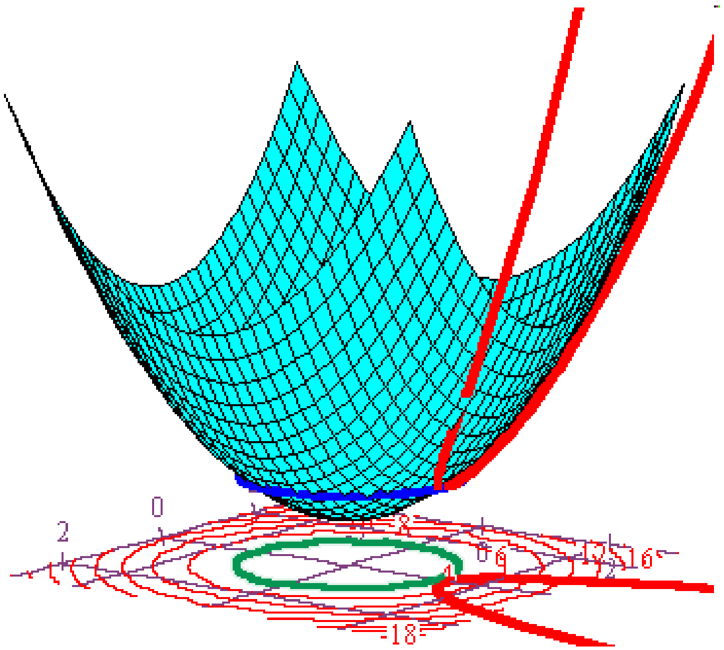
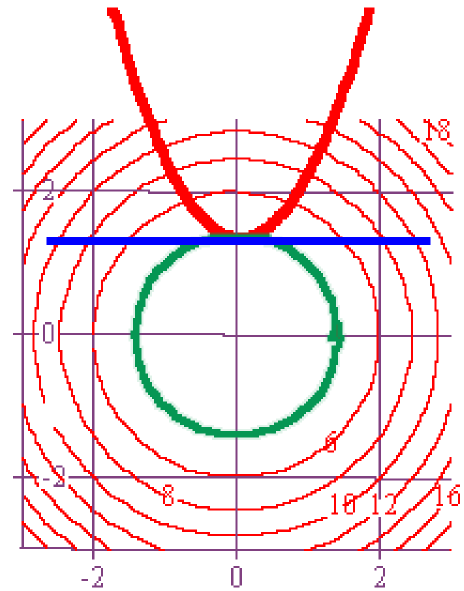




Solve this system of equations and you're done.

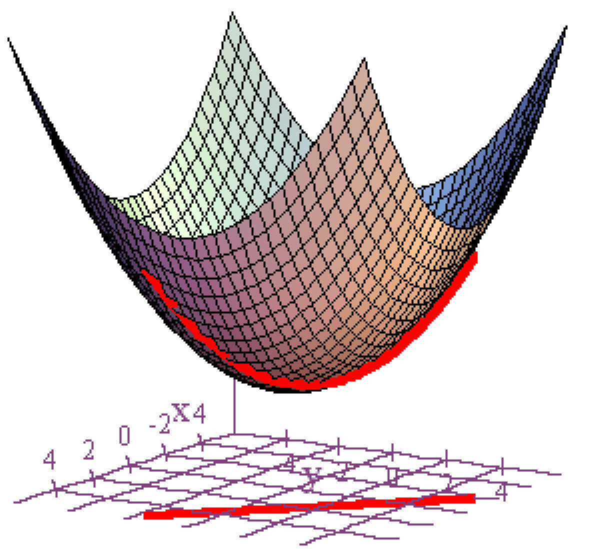
$$z_x = \lambda \cdot g_x$$

$$z_y = \lambda \cdot g_y$$

$$g(x, y) = c$$

Good Luck!

EXAMPLE 1: Suppose $z = f(x,y) = x^2 + y^2 + 20$, and our constraint curve is $x + y = 3$. Find the minimum value of $z = f(x,y)$ on this curve.



$$z = x^2 + y^2 + 20$$
$$g = x + y$$

$$\begin{matrix} z_x = \lambda g_x \\ z_y = \lambda g_y \end{matrix} \Rightarrow \begin{matrix} 2x = \lambda \\ 2y = \lambda \end{matrix} \Rightarrow \begin{matrix} x = \dfrac{\lambda}{2} \\ y = \dfrac{\lambda}{2} \end{matrix}$$

$$x + y = 3 \Rightarrow \frac{\lambda}{2} + \frac{\lambda}{2} = 3 \Rightarrow \lambda = 3 \Rightarrow \begin{matrix} x = 3/2 \\ y = 3/2 \end{matrix}$$

$$f\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{9}{4} + \frac{9}{4} + 20 = \frac{49}{2} = 24.5$$

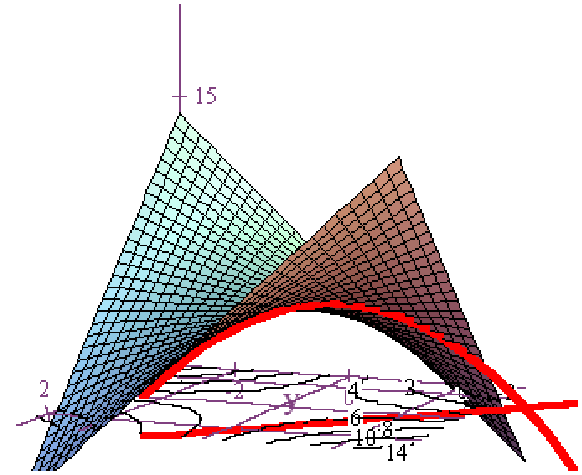
The minimum point is

$$\left(\frac{3}{2}, \frac{3}{2}, \frac{49}{2}\right) = (1.5, 1.5, 24.5).$$

EXAMPLE 2: Suppose $z = f(x, y) = xy + 5$, and our constraint curve is $x + y = 2$. Find the maximum value of $z = f(x, y)$ on this curve.

$$z = xy + 5$$

$$g = x + y$$



$$\begin{matrix} z_x = \lambda g_x \\ z_y = \lambda g_y \end{matrix} \Rightarrow \begin{matrix} y = \lambda \\ x = \lambda \end{matrix} \Rightarrow x = y$$

$$x + y = 2 \Rightarrow x + x = 2 \Rightarrow 2x = 2 \Rightarrow \begin{matrix} x = 1 \\ y = 1 \end{matrix}$$

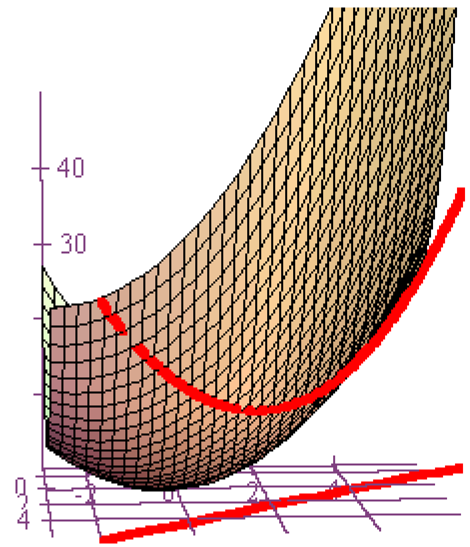
$$f(1,1) = 1 \cdot 1 + 5 = 6$$

The maximum point is $(1,1,6)$.

EXAMPLE 3: Suppose $z = f(x,y) = x^2 + xy + y^2$, and our constraint curve is $x + y = 4$. Find the minimum value of $z = f(x,y)$ on this curve.

$$z = x^2 + xy + y^2$$
$$g = x + y$$

$$\begin{matrix} z_x = \lambda g_x \\ z_y = \lambda g_y \end{matrix} \Rightarrow \begin{matrix} 2x + y = \lambda \\ x + 2y = \lambda \end{matrix} \Rightarrow 2x + y = x + 2y$$
$$\Rightarrow x - y = 0$$



$$\begin{matrix} x - y = 0 \\ x + y = 4 \end{matrix} \Rightarrow 2x = 4 \Rightarrow x = 2 \Rightarrow \begin{matrix} x = 2 \\ y = 2 \end{matrix}$$

$$f(2,2) = 2^2 + 2 \cdot 2 + 2^2 = 12$$

The minimum point is $(2, 2, 12)$.

EXAMPLE 4: A manufacturer has an order for 1000 ultra-deluxe time machines with built-in MP3 player. Suppose the units are manufactured in two different locations with x representing the number of units produced in one location and y the number of units produced in the other. If the total cost of production is given by $z = C(x, y) = x^2 + 10x + 0.50y^2 + 12y - 10,000 \ dollars,$ find the values of x and y that will minimize the costs, and find the minimum cost.

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$$x + y = 1000$$

$$g = x + y$$

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$$x + y = 1000 \qquad z_x = 2x + 10 \qquad g_x = 1$$

$$g = x + y \qquad z_y = y + 12 \qquad g_y = 1$$

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$$x + y = 1000 \qquad z_x = 2x + 10 \qquad g_x = 1$$

$$g = x + y \qquad z_y = y + 12 \qquad g_y = 1$$

$$\begin{array}{l} z_x = \lambda g_x \\ z_y = \lambda g_y \end{array} \Rightarrow \begin{array}{l} 2x + 10 = \lambda \\ y + 12 = \lambda \end{array} \Rightarrow \begin{array}{l} x = \dfrac{\lambda - 10}{2} \\ y = \lambda - 12 \end{array} \Rightarrow \frac{\lambda - 10}{2} + \lambda - 12 = 1000$$

$$\Rightarrow \lambda - 10 + 2\lambda - 24 = 2000 \Rightarrow 3\lambda = 2034 \Rightarrow \lambda = 678$$

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$$g = x + y \qquad z_y = y + 12 \qquad g_y = 1$$

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$$\Rightarrow \lambda - 10 + 2\lambda - 24 = 2000 \Rightarrow 3\lambda = 2034 \Rightarrow \lambda = 678$$

$$x = \frac{678 - 10}{2} = 334$$

$$y = 678 - 12 = 666$$

$$z = \$334{,}666$$