ROW OPERATIONS

Suppose we have a system of equations like the one below. Then there are things you can do to the equations that won't alter the solutions to the system.

$$x + 2y = 5$$
$$2x - y = 0$$

For example, you can interchange two equations.

$$\begin{array}{l} x+2y=5 \\ 2x-y=0 \end{array} \xrightarrow{R_1 \leftrightarrow R_2} \begin{array}{l} 2x-y=0 \\ x+2y=5 \end{array}$$

You can multiply an equation by a non-zero number.

$$\begin{aligned} x + 2y &= 5 \\ 2x - y &= 0 \end{aligned} \xrightarrow{-2R_1 \rightarrow R_1} \begin{aligned} -2x - 4y &= -10 \\ 2x - y &= 0 \end{aligned}$$

And you can add a multiply of one equation to another.

$$\begin{aligned} x + 2y &= 5 \\ 2x - y &= 0 \end{aligned} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{aligned} x + 2y &= 5 \\ -5y &= -10 \end{aligned}$$

These kinds of operations performed on a system of equations are called row operations, and they can be used to help us systematically solve the system.

Row Operations:

1. **Interchange two rows.**
2. **Multiply a row by a nonzero number.**
3. **Add a multiple of one row to another.**

Usually we write the system of equations as an augmented matrix and then perform the row operations as follows:

$$x + 2y = 5$$

$$2x - y = 0$$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right]$$

At this point the matrix is in what we call *row echelon form (ref)*, and it wouldn't be hard to write this result as a system of equations and finish solving for *x* and *y*.

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right]$$

$$x + 2y = 5$$

$$y = 2$$

However, we're going to take it one step further to *reduced row echelon form (rref)*. This form will have ones on the diagonal and zeros above and below the diagonal. The *ref* form has zeros only below the diagonal.

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2\rightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2\rightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{-2R_2+R_1\rightarrow R_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

Now that the system is in *rref* form, we can immediately see what the solutions are.

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$$x + 2y = 5$$
$$2x - y = 0$$

$$x = 1$$
$$y = 2$$

And that's all there is to it!

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$$x + 2y = 5$$
$$2x - y = 0$$

$$x = 1$$
$$y = 2$$