The Second Partials Test

Definition: Let (a,b) be a point contained in an open region R on which a function $z = f(x,y)$ is defined. Then (a,b) is a critical point if any of the following conditions are true:

1. $z_x(a,b) = 0 = z_y(a,b)$
2. $z_x(a,b)$ does not exist
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Theorem: If $z = f(x,y)$ has a local maximum or a local minimum at a point (a,b) contained within an open region R on which $z = f(x,y)$ is defined, then (a,b) is a critical point.

Second Partials Test: Suppose $z = f(x, y)$ has continuous second partial derivatives on an open region containing a point (a,b) such that $z_x(a,b) = 0 = z_y(a,b),$ and let

$$D = D(a,b) = \begin{vmatrix} z_{xx}(a,b) & z_{xy}(a,b) \\ z_{yx}(a,b) & z_{yy}(a,b) \end{vmatrix} = z_{xx}(a,b)z_{yy}(a,b) - z_{xy}(a,b)z_{yx}(a,b).$$

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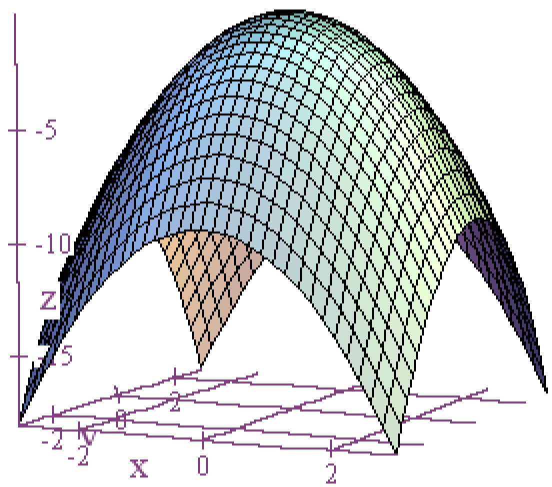
Then:

1. If $D > 0$ and $z_{xx}(a,b) > 0$, $f(a,b)$ is a local minimum.
2. If $D > 0$ and $z_{xx}(a,b) < 0$, $f(a,b)$ is a local maximum.
3. If $D < 0$, $\left(a,b, f(a,b)\right)$ is a saddle point.
4. If $D = 0$, the test is inconclusive.

Example 1: $z = f(x, y) = -x^2 - y^2$



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$$D(0,0) = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 - 0 = 4 > 0$$

$$z_{xx}(0,0) = -2 < 0$$

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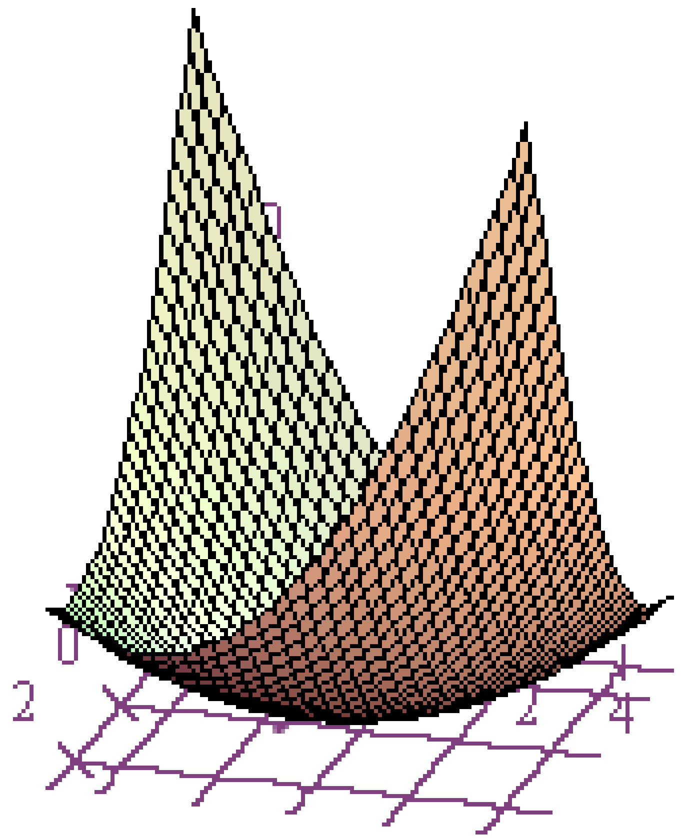
Therefore, $f(0,0) = 0$ is a local maximum,

and $(0,0,0)$ is a maximum point.

Example 2: $z = f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$



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$$D(-1,1) = \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 8 - 4 = 4 > 0$$

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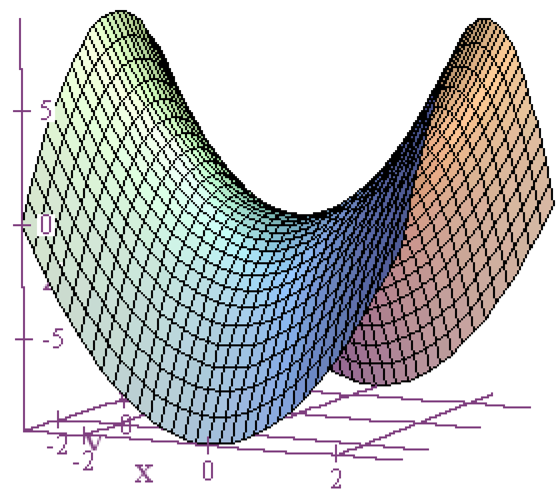
$$z_{xx}(-1,1) = 4 > 0$$

Therefore, $f(-1,1) = -4$ is a local minimum,

and $(-1,1,-4)$ is a minimum point.

Example 3: $z = f(x, y) = x^2 - y^2$



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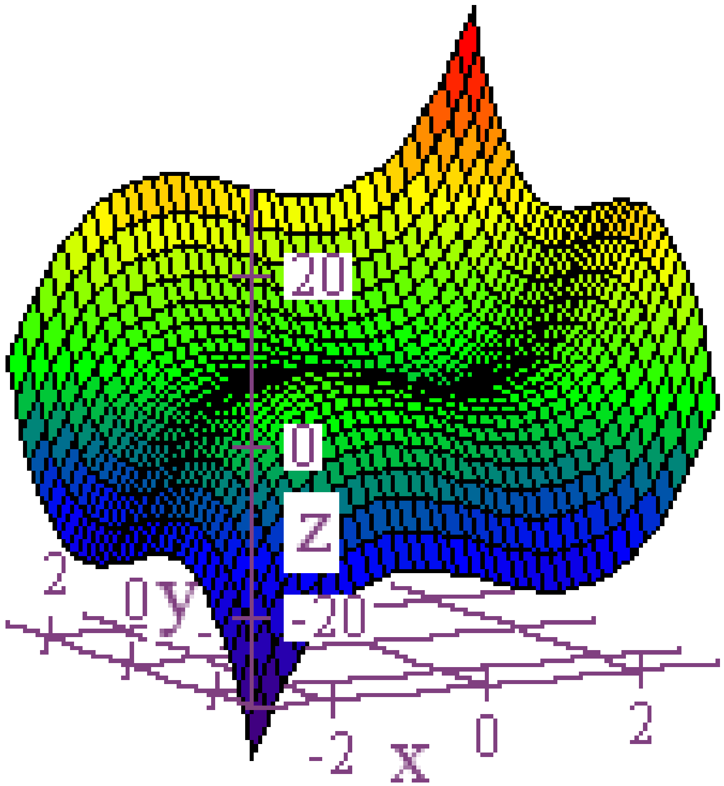
$\Rightarrow$ saddle point

Therefore, $f(0,0) = 0$,

and $(0,0,0)$ is a saddle point.

Try it now with $z = f(x, y) = x^3 - 3x + y^3 - 3y$!



Also try $z = f(x,y) = x^4 - y^4$, $z = f(x,y) = x^4 + y^4$, and $z = -x^4 - y^4$.