DIVERGENCE THEOREM IN THREE DIMENSIONS - ANSWERS

In each problem below you are given a volume V bounded by a surface S along with a vector field F. If N is an outward pointing unit normal vector, then the outward flux across the surface created by the vector field F is, by the Divergence (Gauss') Theorem, equal to $\iint_S \vec{F} \bullet N dS = \iiint_V div\, \vec{F}\, dV = \iiint_V \nabla \bullet F\, dV$. Evaluate this integral for each problem below

1. V is the solid ball with surface S defned by $x^2 + y^2 + z^2 = 1$.
$F = x\hat{i} + y\,\hat{j} + z\,\hat{k}$

$$\iint_S \vec{F} \bullet N dS = \iiint_V \nabla \bullet F\, dV = \iiint_V (1+1+1)\, dV = 3\iiint_V dV = 3 \cdot \frac{4\pi}{3} = 4\pi$$

2. V is the cube defned by $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$, and S is the surface of the cube.
$F = x^2\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

$$\iint_S \vec{F} \bullet N dS = \iiint_V \nabla \bullet F\, dV = \int_0^1\int_0^1\int_0^1 (2x+2)\, dxdydz = \int_0^1\int_0^1 x^2 + 2x \Big|_0^1 dydz$$

$$= \int_0^1\int_0^1 3\, dydx = \int_0^1 3y \Big|_0^1 dz = \int_0^1 3\, dz = 3z \Big|_0^1 = 3$$

3. V is the cylinder defned by $-1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, 0 \le z \le 4$, (bounded below by $z = 0$ and above by $z = 4$), and S is the surface of the cylinder.
$F = y\,\hat{i} - x\,\hat{j}$

$$\iint_S \vec{F} \bullet N dS = \iiint_V \nabla \bullet F\, dV = \iiint_V 0\, dV = 0$$

4. V is the solid bounded by the xy-plane and the hemisphere $z=\sqrt{4-x^2-y^2}$, and S is the surface of V.

$F = x^3\,\hat{i} + y^3\,\hat{j} + z^3\,\hat{k}$

$$\iint_S \vec{F}\bullet N dS = \iiint_V \nabla\bullet F\, dV = \iiint_V (3x^2+3y^2+3z^2)\, dV$$

$$= \iiint_V 3(x^2+y^2+z^2)\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} (3\rho^2)\cdot\rho^2 \sin\varphi\, d\rho d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} 3\rho^4 \sin\varphi\, d\rho d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi/2} \frac{3\rho^5}{5}\sin\varphi\Big|_0^2 d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi/2} \frac{96}{5}\sin\varphi\, d\varphi d\theta$$

$$= \int_0^{2\pi} -\frac{96}{5}\cos\varphi\Big|_0^{\pi/2} d\theta = \int_0^{2\pi}\frac{96}{5}\, d\theta = \frac{96\theta}{2}\Big|_0^{2\pi} = \frac{192\pi}{5}$$

5. V is the solid bounded by the portion in the first octant of the cylinder (including top and bottom) defined by $x^2+y^2=1$ as z varies from 0 to 1, and S is the surface of V.

$F = x^2\,\hat{i}$

$$\iint_S \vec{F}\bullet N dS = \iiint_V \nabla\bullet F\, dV = \iiint_V 2x\, dV$$

$$= \int_0^{\pi/2}\int_0^{1}\int_0^{1} 2r\cos\theta\cdot r\, dz dr d\theta = \int_0^{\pi/2}\int_0^{1}\int_0^{1} 2r^2\cos\theta\, dz dr d\theta$$

$$= \int_0^{\pi/2}\int_0^{1} 2r^2\cos\theta\cdot z\Big|_0^1 dr d\theta = \int_0^{\pi/2}\int_0^{1} 2r^2\cos\theta\, dr d\theta$$

$$= \int_0^{\pi/2} \frac{2r^3}{3}\cos\theta\Big|_0^1 d\theta = \int_0^{\pi/2}\frac{2}{3}\cos\theta\, d\theta = \frac{2}{3}\sin\theta\Big|_0^{\pi/2} = \frac{2}{3}$$