STOKES' THEOREM IN THREE DIMENSIONS - ANSWERS

In each problem below you are given a surface *S*, defined by $z = f(x,y)$, over a region *R*, defined by the given limits on *x* and *y*. Let $C_R$ be the boundary of the region *R*, oriented counterclockwise, and let *C* be the corresponding boundary on the surface *S*, also oriented counterclockwise. Then if *F* is a vector field and *N* is the upward pointing unit normal vector for the surface *S*, use the higher dimensional version of Stokes' Theorem, $\int_C \vec{F}\bullet d\vec{r} = \iint_S \left(curl\, F \bullet N\right) dS$, to measure the circulation around the curve *C* that is caused by the vector field *F*.

Consider $z = f(x,y)$ as a level surface for a function $g = -f(x,y) + z$. Then $N = \dfrac{\nabla g}{\|\nabla g\|}$ will be an upward pointing unit normal to the surface $z = f(x,y)$, and we will have

$$\int_C \vec{F}\bullet d\vec{r} = \iint_S \left(curl\, F \bullet N\right) dS = \iint_R \left[ \left(\nabla \times F\right)\bullet \frac{\nabla g}{\|\nabla g\|} \right] \cdot \|\nabla g\|\, dA = \iint_R \left[ \left(\nabla \times F\right)\bullet \nabla g \right] dA .$$

1. $S: z = -x^2 - y^2 + 4$
   $R: 0 \le x \le 1,\ 0 \le y \le 2$
   $F = z\hat{i} + x\hat{j} + y\hat{k}$

$$g = x^2 + y^2 + z$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} + \hat{k}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

$$\int_C \vec{F}\bullet d\vec{r} = \iint_R \left[ \left(\nabla \times F\right)\bullet \nabla g \right] dA = \int_0^1\int_0^2 \left[ \left(\hat{i} + \hat{j} + \hat{k}\right)\bullet\left(2x\hat{i} + 2y\hat{j} + \hat{k}\right) \right] dydx$$

$$= \int_0^1\int_0^2 \left(2x + 2y + 1\right) dydx = \int_0^1 2xy + y^2 + y \Big|_0^2 dx = \int_0^1 (4x+6)\, dx$$

$$= 2x^2 + 6x\Big|_0^1 = 2 + 6 = 8$$

$S: z = y$

2. $R: 0 \le x \le 1,\ 0 \le y \le 1 - x$

$F = -3y^2\,\hat{i} + 4z\,\hat{j} + 6x\,\hat{k}$

$g = -y + z$

$\nabla g = -\hat{j} + \hat{k}$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -3y^2 & 4z & 6x \end{vmatrix} = -4\hat{i} - 6\,\hat{j} + 6y\,\hat{k}$$

$$\int_C \vec{F} \bullet d\vec{r} = \iint_R \left[ (\nabla \times F) \bullet \nabla g \right] dA = \int_0^1 \int_0^{1-x} \left[ \left( -4\hat{i} - 6\,\hat{j} + 6y\,\hat{k} \right) \bullet \left( -\hat{j} + \hat{k} \right) \right] dydx$$

$$= \int_0^1 \int_0^{1-x} (6 + 6y)\, dydx = \int_0^1 6y + 3y^2 \Big|_0^{1-x} dx = \int_0^1 \left[ 6(1-x) + 3(1-x)^2 \right] dx$$

$$= \int_0^1 (6 - 6x + 3 - 6x + 3x^2)\, dx = \int_0^1 (3x^2 - 12x + 9)\, dx$$

$$= x^3 - 6x^2 + 9x \Big|_0^1 = 1 - 6 + 9 = 4$$

$S: z = x^2 - y^2$

3. $R: -1 \le x \le 1,\ -\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$

$F = x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k}$

$g = -x^2 + y^2 + z$

$\nabla g = -2x\,\hat{i} + 2y\,\hat{j} + \hat{k}$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = \vec{0}$$

$$\int_C \vec{F} \bullet d\vec{r} = \iint_R \left[ (\nabla \times F) \bullet \nabla g \right] dA = \iint_R \left[ \left( 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} \right) \bullet \left( -2x\,\hat{i} + 2y\,\hat{j} + \hat{k} \right) \right] dA$$

$$= \iint_R 0\, dA = 0$$

4. $S: z = 1 - x^2 - y^2$
$R: -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$
$F = z\,\hat{i} + x\,\hat{j} + y\,\hat{k}$
(NOTE: On this problem, let $C_R$ and $C$ be oriented clockwise. This means that your unit normal $N$ will be pointing downward instead of upward.)

$g = 1 - x^2 - y^2 - z$
$\nabla g = -2x\,\hat{i} - 2y\,\hat{j} - \hat{k}$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

$$\int_C \vec{F} \bullet d\vec{r} = \iint_R \left[ \left( \nabla \times F \right) \bullet \nabla g \right] dA = \iint_R \left[ \left( \hat{i} + \hat{j} + \hat{k} \right) \bullet \left( -2x\,\hat{i} - 2y\,\hat{j} - \hat{k} \right) \right] dA$$
$$= \iint_R \left( -2x - 2y - 1 \right) dA = \int_0^{2\pi} \int_0^1 \left( -2r\cos\theta - 2r\sin\theta - 1 \right) \cdot r\, dr d\theta$$
$$= \int_0^{2\pi} \int_0^1 \left( -2r^2\cos\theta - 2r^2\sin\theta - r \right) dr d\theta = \int_0^{2\pi} -\frac{2r^3}{3}\cos\theta - \frac{2r^3}{3}\sin\theta - \frac{r^2}{2}\Bigg|_0^1 d\theta$$
$$= \int_0^{2\theta} \left( -\frac{2}{3}\cos\theta - \frac{2}{3}\sin\theta - \frac{1}{2} \right) d\theta = -\frac{2}{3}\sin\theta + \frac{2}{3}\cos\theta - \frac{\theta}{2}\Bigg|_0^{2\pi} = -\pi$$

$S: z = \frac{1}{2}\sqrt{1 - x^2 - y^2}$

5. $R: -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$F = x^2\,\hat{i} + y^2\,\hat{j} + z^2 \tan xy\,\hat{k}$

(HINT: Use a simpler surface with the same bounding curve.)

Use $z = 0$

$g = z$

$\nabla g = \hat{k}$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \tan xy \end{vmatrix} = (xz^2 \sec^2 xy)\,\hat{i} - (yz^2 \sec^2 xy)\,\hat{j}$$

$$\int_C \vec{F} \bullet d\vec{r} = \iint_R \left[ (\nabla \times F) \bullet \nabla g \right] dA = \iint_R \left[ \left( (xz^2 \sec^2 xy)\,\hat{i} - (yz^2 \sec^2 xy)\,\hat{j} \right) \bullet \left( \hat{k} \right) \right] dA$$

$$= \iint_R 0\,dA = 0$$

6. Let $F = (-6y^2 + 6y)\hat{i} + (x^2 - 3z^2)\,\hat{j} - x^2\,\hat{k}$, and use Stokes' Theorem to show that the work done by *F* along any simple closed curve contained in the plane $x + 2y + z = 1$ is equal to zero.

First we need to find the curl of *F*.

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -6y^2 + 6y & x^2 - 3z^2 & -x^2 \end{vmatrix} = 6z\,\hat{i} + 2x\,\hat{j} + (2x + 12y - 6)\,\hat{k}.$$ Next, let

$g = x + 2y + z$, and find $\nabla g = \hat{i} + 2\,\hat{j} + \hat{k}$. Hence,

$\left(\nabla \times F\right)\bullet\nabla g = 6z + 4x + 2x + 12y - 6 = 6z + 6x + 12y - 6$. If our curve is in the plane $z = -x - 2y + 1$, then

$\left(\nabla \times F\right)\bullet\nabla g = 6z + 6x + 12y - 6 = 6(-x - 2y + 1) + 6x + 12y - 6$

$= -6x - 12y + 6 + 6x + 12y - 6 = 0.$

Therefore, for any simple closed curve *C* in the give plane to which is applied the given vector field *F*, we have $\int_C F \bullet d\vec{r} = \iint_R \left(\nabla \times F\right)\bullet\nabla g \, dA = 0.$