CHAPTER 11

GREEN'S THEOREM, STOKES' THEOREM, AND THE DIVERGENCE THEOREM

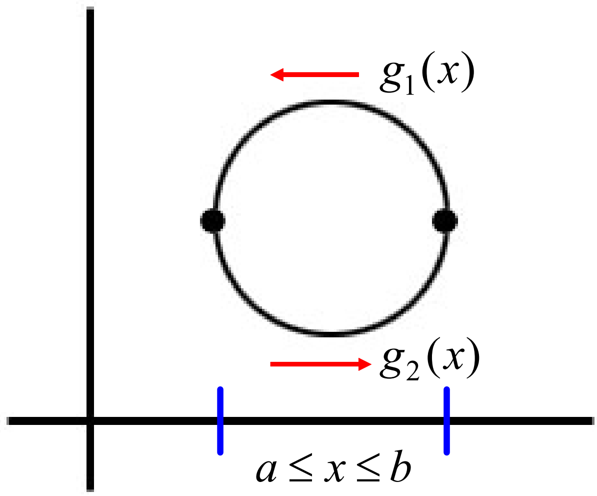
We've already seen one higher dimensional version of the Fundamental Theorem of Calculus, namely the Fundamental Theorem of Line Integrals. Now it's time for another generalization of the FTC, Green's Theorem. Hold on to your hats. This one's a biggie!

Definition: A curve C is closed if its initial point coincides with its terminal point. A closed curve C is simple if it doesn't cross or intersect itself except at its initial and terminal points.

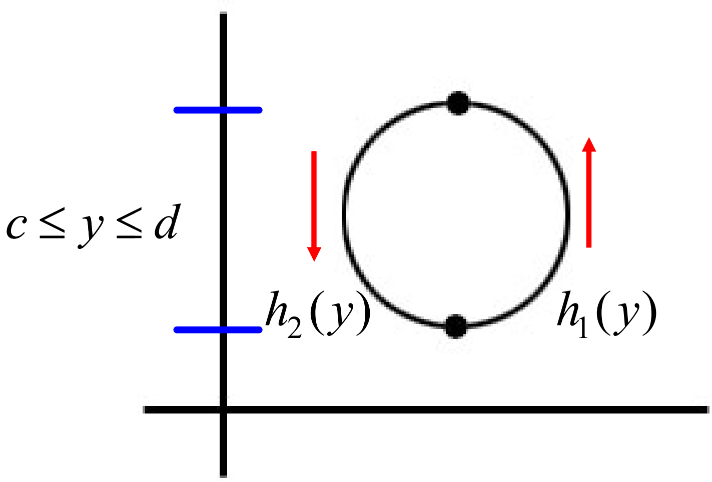
Green's Theorem: Let C be a smooth, simple closed curve that is oriented counterclockwise in the xy-plane, let R be the region bounded by C and let $\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$ be a vector field. If P and Q have continuous partial derivatives on an open region that contains R, then,

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

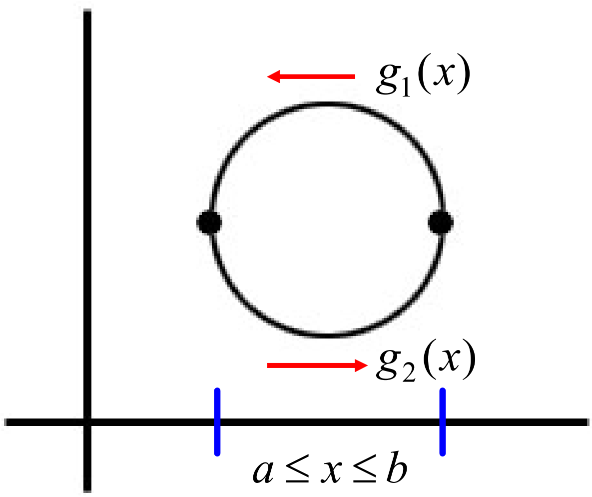
Proof: We'll do just a special case. Thus, suppose our counterclockwise oriented curve C and region R look something like the following:



In this case, we can break the curve into a top part and a bottom part over an interval on the x-axis from a to b, and we can denote the top part by the function $g_1(x)$ and the bottom part by $g_2(x)$. Or, we could just as easily portray x as varying from $h_2(y)$ to $h_1(y)$ as y varies from c to d.



Now let's begin. Suppose the curve below is oriented in the counterclockwise direction and is parametrized by *x*. Suppose also that the top part of our curve corresponds to the function $g_1(x)$ and the bottom part to $g_2(x)$ as indicated in the diagram below.



Then,

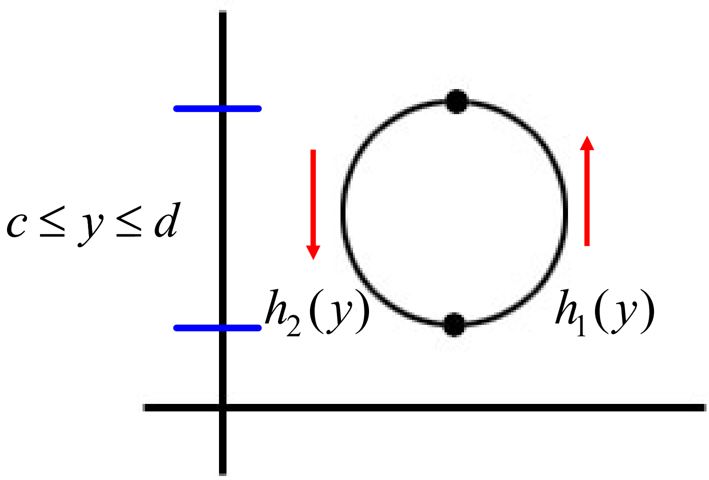
$$\int_C P\,dx = \int_a^b P(x, g_2(x))\,dx + \int_b^a P(x, g_1(x))\,dx$$
$$= \int_a^b P(x, g_2(x))\,dx - \int_a^b P(x, g_1(x))\,dx = \int_a^b P(x, g_2(x)) - P(x, g_1(x))\,dx$$

Notice that our integral of $P(x, g_1(x))$ goes from *b* to *a* since we are traversing the curve in the counterclockwise direction. Also, note the following,

$$-\iint_R \frac{\partial P}{\partial y}\,dA = -\int_a^b \int_{g_2(x)}^{g_1(x)} \frac{\partial P}{\partial y}\,dydx = -\int_a^b P(x, g_1(x)) - P(x, g_2(x))\,dx$$
$$= \int_a^b P(x, g_2(x)) - P(x, g_1(x))\,dx = \int_C P\,dx$$

Therefore, $\int_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\,dA$.

In a similar manner, with respect to the diagram below, we can assert the following,



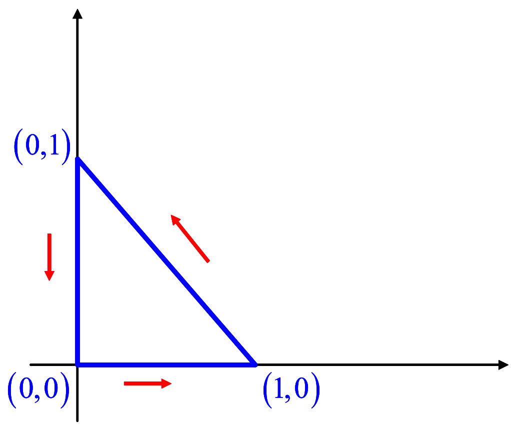
$$\iint_R \frac{\partial Q}{\partial x}\,dA = \int_c^d \int_{h_2(x)}^{h_1(x)} \frac{\partial Q}{\partial x}\,dxdy = \int_c^d Q(h_1(y), y) - Q(h_2(y), y)\,dy$$
$$= \int_c^d Q(h_1(y), y)\,dy + \int_d^c Q(h_2(y), y)\,dy = \int_C Q\,dy$$

Therefore, $\int_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\,dA$, and thus, $\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$. ■

Here is the really remarkable thing that Green's Theorem is saying. We have a region *R* that is bounded by a curve *C*, and Green's Theorem is telling us the value of the double integral of *R* depends entirely upon what happens on the boundary curve *C*. This is exactly what is going on in the Fundamental Theorem of Calculus and the Fundamental Theorem of Line Integrals. The only difference is that in the earlier theorems we dealt with a line instead of a region, and the boundary of the line is represented by its endpoints instead of a curve. Except for the change in dimension, the results are analogous. This is also something that we continue to see as we move higher up into theoretical mathematics. Quite often it's what happens on the boundary that determines things. However, in one respect, this should not surprise us. After all, aren't most things in life defined by their boundary? On physical plane, for example, the boundary of my body defines where I end and the rest of the world begins. Boundaries create the distinctions that result in individual existence. Thus, in a larger sense, these theorems are not so surprising.

Now it's time for an example!

Example 1: Evaluate $\int_C x^4\,dx + xy\,dy$ where C is the positively oriented triangle defined by the line segments connecting $(0,0)$ to $(1,0)$, $(1,0)$ to $(0,1)$, and $(0,1)$ to $(0,0)$.



Solution: By changing the line integral along C into a double integral over R, the problem is immensely simplified.

$$\int_C x^4\,dx + xy\,dy = \int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^1 \int_0^{1-x} (y-0)\,dydx$$

$$= \int_0^1 \frac{y^2}{2}\bigg|_0^{1-x} dx = \int_0^1 \frac{(1-x)^2}{2} dx = \frac{-(1-x)^3}{2\cdot 3}\bigg|_0^1 = \frac{1}{6}$$

Now let's look at another problem which can be greatly simplified by applying Green's Theorem.

Example 2: Evaluate $\int_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$ where C is the circle $x^2 + y^2 = 9$.

Solution: Again, Green's Theorem makes this problem much easier.

$$\int_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy = \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_R \left[\frac{\partial(7x + \sqrt{y^4+1})}{\partial x} - \frac{\partial(3y - e^{\sin x})}{\partial y} \right] dA$$

$$= \iint_R (7-3)\,dA = \int_0^{2\pi}\int_0^3 4r\,dr\,d\theta = \int_0^{2\pi} 2r^2 \Big|_0^3 \, d\theta$$

$$= \int_0^{2\pi} 18\,d\theta = 18\theta \Big|_0^{2\pi} = 36\pi$$

Now let's look at a couple of generalizations of Green's Theorem, namely Stokes' Theorem and the Divergence Theorem. In two dimensions, it's very easy to see that these are both simply different ways of looking at Green's Theorem, and so we'll start with the two dimensional versions of these results. Also, we'll finally see in these theorems why the definitions that we gave for curl and divergence wind up telling us something about the circulation and flux of a vector field along or across a curved path.

Let's begin by supposing that we have a vector field $\vec{F} = P\hat{i} + Q\hat{j}$ and that $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ is a smooth parametrization for a curve *C*. Then Green's Theorem and previous results tells us that,

$$\text{Work } = \int_C \vec{F} \bullet d\vec{r} = \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

However, recall that this same integral is also a measure of the circulation around he curve caused by the vector field. Additionally, notice that the expression $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ is how we defined the scalar component of the curl for a two dimensional vector field. Thus,

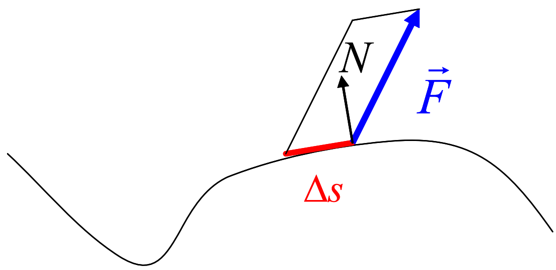
$$\begin{aligned}\text{Circulation } &= \int_C \vec{F} \bullet T\,ds = \int_C \vec{F} \bullet d\vec{r} = \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \iint_R \left(curl\,\vec{F} \right) \bullet \hat{k}\,dA = \iint_R \left(\nabla \times \vec{F} \right) \bullet \hat{k}\,dA\end{aligned}$$

This last result that $\text{Circulation } = \int_C \vec{F} \bullet d\vec{r} = \iint_R \left(curl\,\vec{F} \right) \bullet \hat{k}\,dA$ is known as the two dimensional version of Stoke's Theorem.

To develop the two dimensional version of the Divergence Theorem, recall that if

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ with $a \le t \le b$ is parametrization for C, then the unit tangent is defined as $T(t) = \dfrac{\vec{r}'(t)}{\|r'(t)\|} = \dfrac{x'(t)}{\|r'(t)\|}\hat{i} + \dfrac{y'(t)}{\|r'(t)\|}\hat{j}$, and the unit normal is $N(t) = \dfrac{y'(t)}{\|r'(t)\|}\hat{i} - \dfrac{x'(t)}{\|r'(t)\|}\hat{j}$.

Now consider the diagram below.



What we want to think about is how a force $\vec{F}$ might push material across this curve. In this case, think of $\vec{F}$ as a velocity vector. Additionally, if our change in arc length, Δs, is small, then we can treat our velocity vector as if it were constant over this interval. In this case, the amount of material or flux across this boundary in a unit of time is going to be approximately equivalent to the area of the parallelogram defined by the Δs and the vector $\vec{F}$. To get the height of this parallelogram, we just take the dot product of $\vec{F}$ with the unit normal vector N, and then we multiply this result by the base Δs. In other words, $\textit{flux across } \Delta s = \textit{area} = \left(\vec{F} \bullet N\right)\Delta s$. To now get the total flux across the curve C, we just sum up all the individual fluxes and take a limit as Δs goes to zero. In other words,

$$Flux = \lim_{\Delta s \to 0} \sum \left(\vec{F} \bullet N\right) \Delta s = \int_C \left(\vec{F} \bullet N\right) ds$$

Now, as you might suspect, there are some other ways in which we can write this integral, and it's going to be a direct result of applying Green's Theorem.

$$\begin{aligned}
\text{Flux } &= \int_C \vec{F} \bullet N \, ds = \int_a^b \left(\vec{F} \bullet N\right) \frac{ds}{dt} dt \\
&= \int_a^b \left(P\hat{i} + Q\hat{j}\right) \bullet \left(\frac{y'(t)}{\left\| \vec{r}'(t) \right\|} \hat{i} - \frac{x'(t)}{\left\| \vec{r}'(t) \right\|} \right) \left\| \vec{r}'(t) \right\| dt \\
&= \int_a^b \left(P\hat{i} + Q\hat{j}\right) \bullet \left(\frac{dy}{dt} \hat{i} - \frac{dx}{dt} \right) dt = \int_a^b \left(P \frac{dy}{dt} - Q \frac{dx}{dt} \right) dt \\
&= \int_C P dy - Q dx = \int_C -Q dx + P dy = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA \\
&= \iint_R div \, \vec{F} \, dA = \iint_R \left(\nabla \bullet \vec{F} \right) dA
\end{aligned}$$

If you look at this proof closely, you'll see that it uses a lot of tools that we've developed throughout this book, and you'll also see at the very end why our definition of the divergence does indeed tell us something about flux across a boundary.

The Divergence Theorem is also known as Gauss' Theorem, and below we have a summary of our results for the two dimensional case.

If $\vec{F} = P\hat{i} + Q\hat{j}$ is a vector field and if C is a simple closed, counterclockwise oriented path parametrized by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, and if T is the unit tangent vector and N is the unit normal vector, then:

GREEN'S THEOREM: Work $= \int_C \vec{F} \bullet d\vec{r} = \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

STOKES' THEOREM: Circulation $= \int_C \vec{F} \bullet T\,ds = \int_C \vec{F} \bullet d\vec{r}$

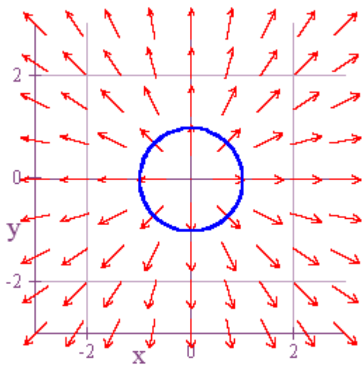
$$= \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R \left(curl\,\vec{F} \right) \bullet \hat{k}\,dA = \iint_R \left(\nabla \times \vec{F} \right) \bullet \hat{k}\,dA$$

GAUSS' THEOREM: Flux $= \int_C \vec{F} \bullet N\,ds == \int_C Pdy - Qdx = \int_C -Qdx + Pdy$

$$= \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_R div\,\vec{F}\,dA = \iint_R \left(\nabla \bullet \vec{F} \right) dA$$

Let's now look at an example or two before moving on.

<u>Example 3:</u> Find the circulation and flux of $\vec{F} = x\hat{i} + y\hat{j}$ with regard to the unit circle below. Assume a positive (counterclockwise) orientation for the curve.

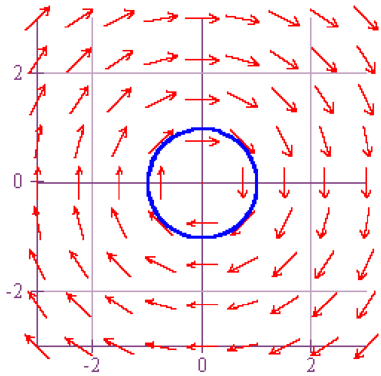


<u>Solution:</u> Our initial suspicion should be that the circulation is zero and that the flux is positive since our vectors are pointing away from the center of the circle. Calculations will confirm this.

$$\text{Circulation } = \int_C \vec{F} \bullet d\vec{r} = \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) dA$$
$$= \iint_R (0-0)\,dA = 0$$

$$\text{Flux } = \int_C \vec{F} \bullet N\,ds = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_R \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right) dA = \iint_R (1+1)\,dA$$
$$= 2\iint_R dA = 2 \cdot (\text{area of the circle}) = 2\pi$$

Example 4: Find the circulation and flux of $\vec{F} = y\hat{i} - x\hat{j}$ with regard to the unit circle below. Assume a positive (counterclockwise) orientation for the curve.



Solution: This time our suspicion should be that the circulation is negative since the vectors suggest a rotation in the clockwise or negative direction, and that the flux is zero.

$$\text{Circulation } = \int_C \vec{F} \bullet d\vec{r} = \int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R \left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y} \right) dA$$
$$= \iint_R (-1-1)\, dA = -2 \iint_R dA = -2 \cdot (\text{area of the circle}) = -2\pi$$

$$\text{Flux } = \int_C \vec{F} \bullet N\, ds = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_R \left(\frac{\partial y}{\partial x} + \frac{\partial(-x)}{\partial y} \right) dA$$
$$= \iint_R (0+0)\, dA = 0$$

Now let's look at Stokes' Theorem in three dimensions. We'll basically do just the case that is easiest to understand. In particular, we'll assume that we have a smooth surface $z = f(x,y)$ that is bounded by a curve *C*, and we'll assume that the domain of $z = f(x,y)$ is a nice, simply connected region *R* in the *xy*-plane bounded by a curve C_R that, of course, gets mapped onto the curve *C* by our function $z = f(x,y)$. We'll also assume that $\vec{F}(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$ is a vector field that has continuous partial derivatives on an open region in $\mathbb{R}^3$ (three dimensional coordinate space) that contains the surface *S*. As usual, continuity prevents anything really unusual, bad, or unexpected from happening, and given these conditions, we can succinctly state our theorem as,

<u>Stokes' Theorem:</u> $\int_C \vec{F} \bullet d\vec{r} = \iint_S \left(curl\, \vec{F} \bullet N \right) dS$.

<u>Proof:</u> Let's talk first about the vector *N*. What the heck is that? We'll as you might suspect, *N* is a unit normal vector, but in this case we're talking about a vector being normal or perpendicular to the surface $z = f(x,y)$. So how do we find our unit normal? Again, we already know one way to do it. Remember that we can write $z = f(x,y)$ as $0 = f(x,y) - z$, and we can consider the surface *S* to be a level surface of the function $g(x,y,z) = f(x,y) - z$. Consequently, as you surely recall, the gradient of

g is $\nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} - \hat{k}$, and ∇g is normal to the surface at any surface point we wish to evaluate it at. Now realize this. If ∇g is perpendicular to the surface, then $-\nabla g$ is also perpendicular to the surface. Right? It just points in the opposite direction as ∇g. Also, $-\nabla g$ is the vector we are going to use because since $-\nabla g = -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}$ has a positive $\hat{k}$ component, it will point upward with respect to our surface, and that is going to be much nicer. Now, how do we get a unit normal vector out of his? Simple! We just divide $-\nabla g$ by its length. Thus,

$$N = \frac{-\nabla g}{\|-\nabla g\|} = \frac{-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}.$$

Now recall that when we did a surface integral a long time ago as an example of a double integral, we saw that $dS = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\,dA$. Consequently,

$$\iint_S \left(curl\,\vec{F} \bullet N\right) dS = \iint_R \left[curl\,\vec{F} \bullet \frac{-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}} \right] \cdot \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\,dA$$

$$= \iint_R \left[curl\,\vec{F} \bullet \left(-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k} \right) \right] dA$$

Now we're making progress! The next step is to remember that

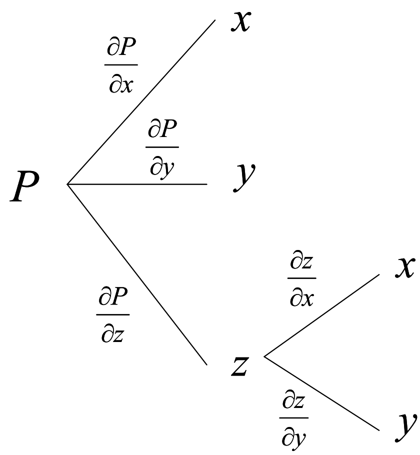
$$curl\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}.$$ Thus,

$$\iint_S \left(Curl\,F \bullet N\right) dS = \iint_R \left[curl\,\vec{F} \bullet \left(-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k} \right) \right] dA$$
$$= \iint_R \left[\left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} \right) \bullet \left(-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k} \right) \right] dA$$
$$= \iint_R \left(-\frac{\partial R}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial P}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$
$$= \iint_R \left(-\frac{\partial z}{\partial x}\frac{\partial R}{\partial y} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Also, $\int_C \vec{F} \bullet d\vec{r} = \int_C P dx + Q dy + R dz$. So, all we have to do is to show that

$$\int_C P dx + Q dy + R dz = \iint_R \left(-\frac{\partial z}{\partial x}\frac{\partial R}{\partial y} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$ Is that too much to ask? Certainly not! Let's begin.

There are two things we need now. First, let's suppose that our curve C_R is nicely parametrized by $x = x(t)$ and $y = y(t)$ for $a \le t \le b$. Then this also provides us with a parametrization for *C* if we let $z = z(t) = f\left(x(t), y(t)\right)$. Second, we're going to have to use some unusual versions of the chain rule, and so let's look at the instructive diagrams below. For the function *P*, our diagram is,



Hence, if we want the derivative of P with respect to y, then we multiply along all the branches that terminate in y and add 'em up. This gives us,

$$\text{derivative of } P \text{ with respect to } y = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}.$$

If you're really sharp, you might notice that there is something a little funny about what I just wrote. In particular, I wrote out the phrase "derivative of P with respect to y" instead of using the symbolic notation $\frac{\partial P}{\partial y}$. However, if I had used this notation, then I would have gotten the equation

$$\frac{\partial P}{\partial y} = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}$$

which would imply that

$$0 = \frac{\partial P}{\partial z}\frac{\partial z}{\partial y},$$

and that is not at all what we want! So what's going on here? Well, basically, our notation is failing us a bit, so let's make up an example and follow it through. Suppose we have,

$$P = y^3 + z^2 \quad \&$$
$$z = y^4$$

Then what we are really trying to say when we write

$$\text{derivative of } P \text{ with respect to } y = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}$$

is,

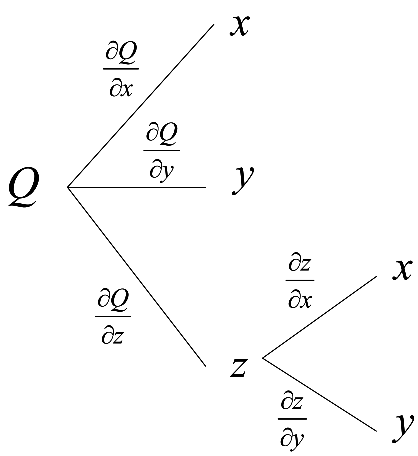
$$\begin{aligned}&\text{derivative of } P \text{ with respect to } y\\ &\quad = \text{derivative of the part that is explicitly written in terms of } y\\ &\qquad + \text{derivative of the part that is explicity written in terms of } z\\ &\qquad\quad \times \text{derivative of } z \text{ with respect to } y\end{aligned}$$

Thus, to find $\frac{\partial P}{\partial y}$ in this example, we first find the derivative of y^3 with respect to y, and then we apply the chain rule to z^2. The end result is,

$$\frac{\partial P}{\partial y} = \frac{\partial (y^3)}{\partial y} + \frac{\partial (z^2)}{\partial z}\frac{\partial z}{\partial y} = 3y^2 + 2z \cdot \frac{\partial (y^4)}{\partial y} = 3y^2 + 2y^4 \cdot 4y^3 = 3y^2 + 8y^7$$

Make sense? Good! Now let's continue with the proof.

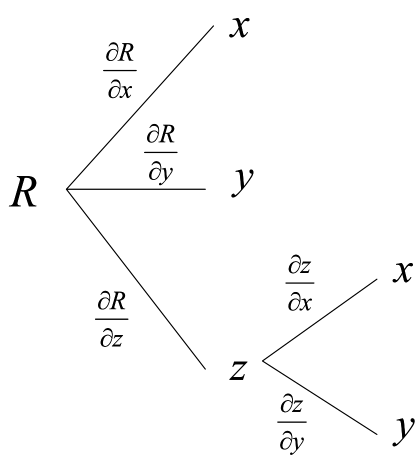
For our function Q, the diagram is,



And,

$$\text{derivative of } Q \text{ with respect to } x = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x}.$$

Finally, the diagram for R is,



Furthermore,

$$\text{derivative of } R \text{ with respect to } x = \frac{\partial R}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}$$

And,

$$\text{derivative of } R \text{ with respect to } y = \frac{\partial R}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}$$

Now we’re ready to rock-n-roll! Just make the above substitutions into the calculations below when the time comes.

$$\int_C \vec{F}\bullet d\vec{r} = \int_C Pdx + Qdy + Rdz = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt$$

$$= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \right) \right] dt = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{\partial z}{\partial x}\frac{dx}{dt} + R\frac{\partial z}{\partial y}\frac{dy}{dt} \right) dt$$

$$= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x} \right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y} \right)\frac{dy}{dt} \right] dt = \int_{C_R} \left(P + R\frac{\partial z}{\partial x} \right) dx + \left(Q + R\frac{\partial z}{\partial y} \right) dy$$

$$= \iint_R \left[\frac{\partial\left(Q + R\frac{\partial z}{\partial y} \right)}{\partial x} - \frac{\partial\left(P + R\frac{\partial z}{\partial x} \right)}{\partial y} \right] dA$$

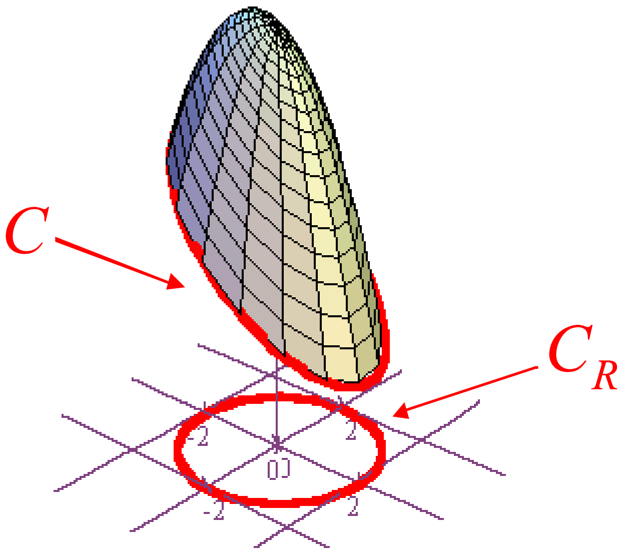
$$= \iint_R \left[\begin{array}{c} \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} \right) + \left(R\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y}\left[\frac{\partial R}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x} \right] \right) \\ -\left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} \right) - \left(R\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial x}\left[\frac{\partial R}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y} \right] \right) \end{array} \right] dA$$

$$= \iint_R \left(\begin{array}{l} \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y}\frac{\partial R}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial R}{\partial z}\frac{\partial z}{\partial x} - \frac{\partial P}{\partial y} - \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} \\ \qquad - R\frac{\partial^2 z}{\partial y \partial x} - \frac{\partial z}{\partial x}\frac{\partial R}{\partial y} - \frac{\partial z}{\partial x}\frac{\partial R}{\partial z}\frac{\partial z}{\partial y} \end{array} \right) dA$$

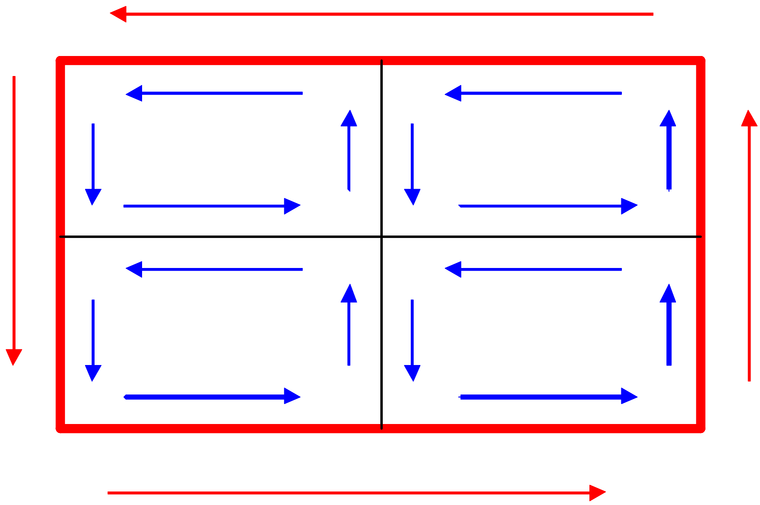
$$= \iint_R \left(-\frac{\partial z}{\partial x}\frac{\partial R}{\partial y} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_S \left(curl\,\vec{F}\bullet N \right) dS \qquad \blacksquare$$

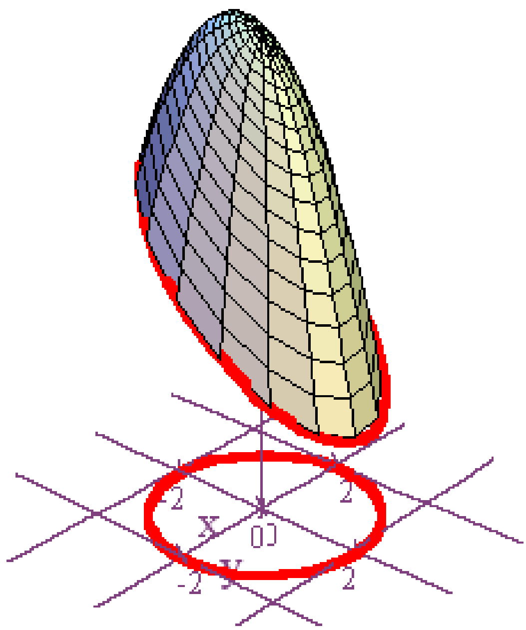
Wow! What a great proof! I'm really stoked! Now let's look at a few pictures in order to get another perspective on what Stokes' Theorem is saying.



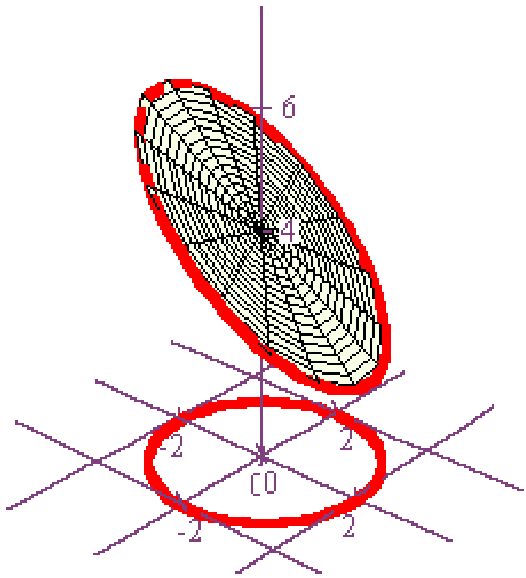
Above is a picture of a surface *S* that is bounded by a curve *C*. Right below *C* there is a corresponding curve C_R in the *xy*-plane that bounds a region *R*. Then the proof of Stokes' Theorem essentially shows us that the circulation integral around *C* is equivalent to a double integral of the curl over *S* which is equivalent to a double integral of the region *R* directly below *S* which is equivalent to a circulation integral around the curve C_R in the *xy*-plane. The bottom line of all these conclusions, though, is simply that the double integral of our curl over *S* is equivalent to the circulation integral around *C*. This is also somewhat easy to intuit from the following diagram.



Suppose we are trying to do a line integral over all the oriented paths represented by the blue arrows. Then the net result is going to be that some of these integrals are going to cancel each other out because for one integral our blue arrow will be pointing in one direction and for another integral our arrow will be pointing in the opposite direction. In fact the only paths that don't cancel are those represented by the red arrows. A similar thing happens with surfaces in three dimensions. In the graph below, if you imagine doing a line integral in the counterclockwise direction around each little polygon that we are using to help depict our surface, then once again lots of things will cancel out, and we'll be left only with a line integral around the bounding curve *C* at the bottom of the surface. What this is basically showing us is that an integral over the surface is approximately equivalent to a sum of integrals over the polygons used to depict that surface which is equivalent, by an extended version of Green's Theorem, to a sum of line integrals around the boundaries of those polygons which in turn reduces to a single line integral around the curve that bounds our surface. Case closed!



Notice, too, that the result of Stokes' Theorem does not depend so much on what our surface looks like. It only depends on the bounding curve. Thus, for a given vector field, the diagram below will give the same result as the one above.



Now let's see if we can do an example. We'll let our surface be the top half of a sphere of radius 1. More precisely, let $z=\sqrt{1-x^2-y^2}$ and let our vector field be $\vec{F}=-y\hat{i}+x\hat{j}+z\hat{k}$. Then the bounding curve C is the unit circle in the xy-plane, and we can think of our surface as a level surface for the function

$g=-f(x,y)+z=-(1-x^2-y^2)^{1/2}+z$. Furthermore,

$$\begin{aligned}\nabla g&=-\frac{1}{2}(1-x^2-y^2)^{-1/2}(-2x)\hat{i}-\frac{1}{2}(1-x^2-y^2)^{-1/2}(-2y)\hat{j}+\hat{k}\\&=\frac{x}{\sqrt{1-x^2-y^2}}\hat{i}+\frac{y}{\sqrt{1-x^2-y^2}}\hat{j}+\hat{k}\end{aligned}.$$

Also,

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ -y & x & z\end{vmatrix}=0\hat{i}-0\hat{j}+(1+1)\hat{k}=2\hat{k}\,.$$

Thus, by Stokes’ Theorem,

$$\int_C \vec{F}\bullet d\vec{r}=\iint_S\left(curl\,\vec{F}\bullet N\right)dS=\iint_R\left(\nabla\times\vec{F}\right)\bullet\nabla g\,dA$$
$$=\iint_R 2\hat{k}\bullet\left(\frac{x}{\sqrt{1-x^2-y^2}}\hat{i}+\frac{y}{\sqrt{1-x^2-y^2}}\hat{j}+\hat{k}\right)dA=\iint_R 2\,dA=2\pi$$

If we try and do this by evaluating the line integral on the left side more directly, then we begin with $\int_C \vec{F}\bullet d\vec{r}=\int_C Pdx+Qdy+Rdz$ where C is the unit circle that may be parametrized by,

$$\begin{aligned}&x=\cos t\\&y=\sin t\\&z=0\\&0\le t\le 2\pi\end{aligned}$$

And,

$$\begin{aligned}&P=y=-\sin t\\&Q=x=\cos t\\&R=1\end{aligned}$$

Additionally,

$$\frac{dx}{dt} = -\sin t$$
$$\frac{dy}{dt} = \cos t$$

Thus,

$$\int_C \vec{F} \bullet d\vec{r} = \int_C P dx + Q dy + R dz = \int_0^{2\pi} P\frac{dx}{dt}dt + Q\frac{dy}{dt}dt + R\frac{dz}{dt}dt$$
$$= \int_0^{2\pi} (-\sin t)(-\sin t) + (\cos t)(\cos t) + 0 \, dt$$
$$= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} dt = t \Big|_0^{2\pi} = 2\pi$$

So there you go. We get the same answer either way.

Now let's take a look at the Divergence Theorem.

<u>Divergence Theorem:</u> Let V be a solid region bounded by a closed surface S and let N be a unit normal vector pointing outward from the solid V. If $\vec{F} = P\hat{i} + Q\hat{j} + Rk$ is a vector field who component functions have continuous partial derivatives throughout V, then $\iint_S \vec{F} \bullet N \, dS = \iiint_V \left(\nabla \bullet \vec{F}\right) dV$.

Proof: We want to show that

$\iint_S \vec{F} \bullet N\, dS = \iint_S \left(P\hat{i} \bullet N + Q\hat{j} \bullet N + Rk \bullet N \right) dS = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV$. Furthermore, it will suffice to show that,

$$\iint_S \left(P\hat{i} \bullet N \right) dS = \iiint_V \frac{\partial P}{\partial x} dV,$$

$$\iint_S \left(Q\hat{j} \bullet N \right) dS = \iiint_V \frac{\partial Q}{\partial y} dV, \text{ and}$$

$$\iint_S \left(R\hat{k} \bullet N \right) dS = \iiint_V \frac{\partial R}{\partial z} dV$$

We'll prove just the last equality, $\iint_S \left(R\hat{i} \bullet N \right) dS = \iiint_V \frac{\partial R}{\partial z} dV$, since the other proofs are similar. Of course, what a professor generally means when he or she says this is,

1. I've never even tried to prove the other cases.
2. I don't even know if I can do the other cases.
3. Oh, please, oh, please, don't make me try and do the other cases!!!

But, continuing on, let's suppose our solid has a top surface S_2 defined by $z = f_2(x, y)$, and a bottom surface S_1 defined by $z = f_1(x, y)$, and maybe some vertical sides. If we set $g_2 = -f(x, y) + z$ and $g_1 = f(x, y) - z$, then ∇g_2 and ∇g_1 will both be outward pointing vectors that are perpendicular to our surface *S*. We don't need to really worry about the vertical sides because for any unit normal N at such a side point we automatically have $R\hat{k}\bullet N = 0$. With this setup we now have,

$$\iint_S \left(R\hat{k}\bullet N\right)dS = \iint_{S_2}\left(R\hat{k}\bullet N\right)dS + \iint_{S_1}\left(R\hat{k}\bullet N\right)dS$$
$$= \iint_R R(x,y,f_2(x,y))\hat{k}\bullet\left[-\frac{\partial f}{\partial x}\hat{\imath} - \frac{\partial f}{\partial y}\hat{\jmath} + \hat{k}\right]dA + \iint_R R(x,y,f_1(x,y))\hat{k}\bullet\left[\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} - \hat{k}\right]dA$$
$$= \iint_R R(x,y,f_2(x,y)) - R(x,y,f_1(x,y))\ dA$$
$$= \iint_R\left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial R}{\partial z}dz\right]dA = \iiint_V \frac{\partial R}{\partial z}dV. \qquad \blacksquare$$

And there you go! Slam dunk!! The proof is similar for the other parts (so they say), and when you add everything together you get $\iint_S \vec{F}\bullet N\,dS = \iiint_V\left(\nabla\bullet\vec{F}\right)dV$.

And now it's time to make an example of this theorem! We'll use the same volume and vector field we used last time, but this time we'll use the Divergence Theorem to find the flux across the boundary. So once again, let our surface be the top half of a sphere of radius 1, $z = \sqrt{1-x^2-y^2}$, and let our vector field be $\vec{F} = -y\hat{\imath} + x\hat{\jmath} + z\hat{k}$. Then the corresponding region R in the xy-plane is the unit circle. And as before, we can think of our surface as a level surface for the function $g = -f(x,y) + z = -(1-x^2-y^2)^{1/2} + z$. Now, using the Divergence Theorem, we have,

$$\iint_S \vec{F}\bullet N\,dS = \iiint_V \left(\nabla\bullet\vec{F}\right)dV = \iiint_V dV = \frac{2\pi}{3}$$

If we now try to integrate our surface integral a little more directly, then we get,

$$\iint_S \vec{F}\bullet N\,dS = \iint_R \left(\vec{F}\bullet\nabla g\right)dA = \iint_R \left(\frac{-xy}{\sqrt{1-x^2-y^2}} + \frac{xy}{\sqrt{1-x^2-y^2}} + z\right)dA$$

$$= \iint_R z\,dA = \iint_R \sqrt{1-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\,rdrd\theta = \left[\text{set } u = 1-r^2,\ du = -2rdr\right]$$

$$= -\frac{1}{2}\int_0^{2\pi}\int_1^0 u^{1/2}\,dud\theta = -\frac{1}{2}\int_0^{2\pi}\left[\left.\frac{2u^{3/2}}{3}\right|_1^0\right]d\theta = \int_0^{2\pi}\frac{1}{3}d\theta = \left.\frac{\theta}{3}\right|_0^{2\pi} = \frac{2\pi}{3}$$

We got the same answer twice, so it must be right. Adios, for now. It's been a pleasure!