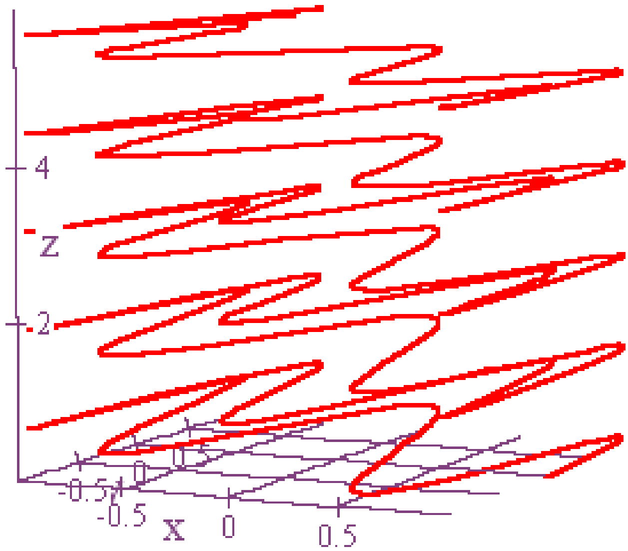
CHAPTER 3

PARAMETRIC EQUATIONS FOR CURVES IN SPACE

At this point we've talked an awful lot about graphs of functions of two variables and also about graphing surfaces in both cylindrical and spherical coordinates, and we've looked at an incredible number of examples of the kinds of surfaces that we can generate. However, sometimes it's not really a surface at all that we want to describe. Sometimes we just want to describe a curve or path that something like a baseball might travel when thrown. How do we do that? Well, the easiest way is usually to express the ball's location in space as a function of time. When we do this, we think of time, *t*, as a parameter for determining location of our object in coordinates (x, y, z). Consequently, we need our variables *x*, *y*, and *z* to all be expressed as functions of *t*. When we specify all three functions as well as the range of values for *t*, then we call the result our parametric equations for the curve that, in this case, our baseball will travel.

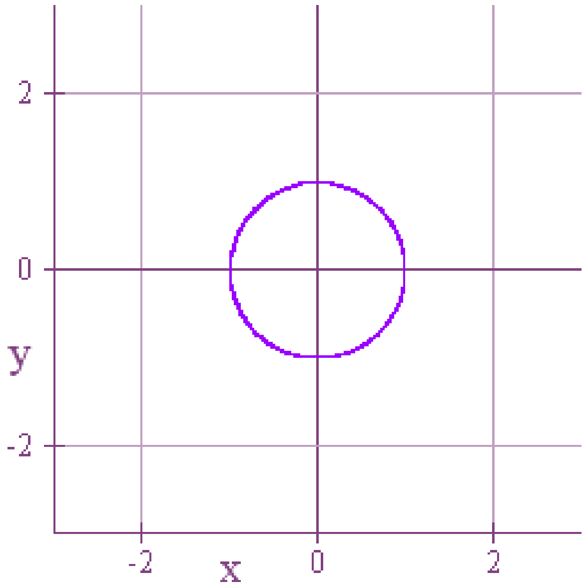
Now let's look at a simple example. Below are some parametric equations followed by the curve they produce.

$$x = \cos(t)$$
$$y = \sin(5t)$$
$$z = \frac{t}{5}$$
$$0 \le t \le 30$$



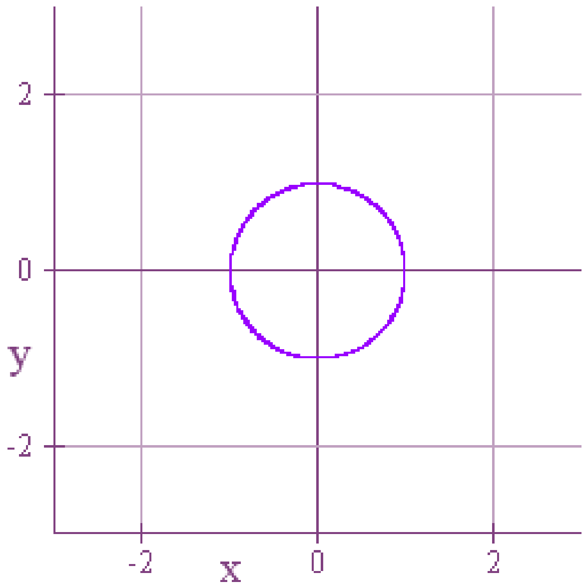
Well, that looks pretty disgusting! The good news is that we won't really need to know how to do very many parametrizations. In fact, there are only three things we'll need to know how to do well: (1) how to construct parametric equations for a line, (2) how to construct parametric equations for a circle, and (3) how to construct parametric equations for a cross-section of certain types of planes with a surface. Let's begin with the circle.

Fotunarely, you probably already know how to parametrize a circle – you just don't know that you know. Nevertheless, if you think back to trigonometry and polar coordinates, then you'll recall that for a circle of radius 1 with center at the origin, we have $x = \cos\theta$ and $y = \sin\theta$ where $0 \le \theta \le 2\pi$. We can also think of these equations as parametric equations where the parameter is θ. If we graph the corresponding curve in 2-dimensions, then we get our unit circle.



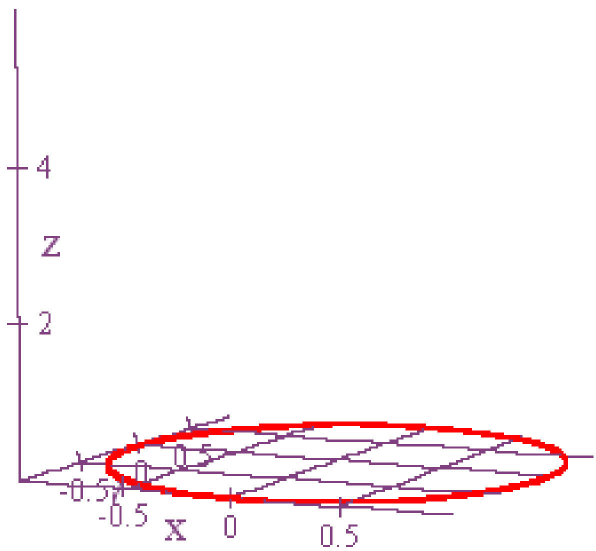
In this case, if we start with $\theta = 0$ and end with $\theta = 2\pi$, then we'll trace our circle in the counterclockwise direction both starting and ending at the point $(1,0)$. Much later on, we'll designate the counterclockwise direction as the positive direction, and we'll be very concerned about which direction our curve is traced in. For now, however, it won't be that much of a concern for us. One of the things we do want to take note of at this point, though, is that there will always exist an infinite number of parametrizations for any particular curve. For example, if in our parametric equations above, we replace θ by $\theta/2$ and change the range for θ to $0 \le \theta \le 4\pi$, then the end result is the same.

$$x = \cos(\theta/2)$$
$$y = \sin(\theta/2)$$
$$0 \le \theta \le 4\pi$$



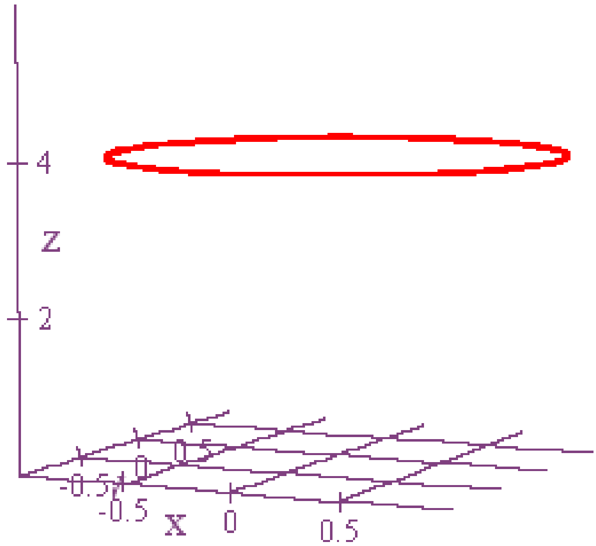
If we want to graph this circle in 3-dimensions but keep it in the *xy*-plane, then all we have do is to add the coordinate $z = 0$. By the way, at this point I'm going to switch to using *t* for the parameter.

$$x = \cos(t)$$
$$y = \sin(t)$$
$$z = 0$$
$$0 \le t \le 2\pi$$



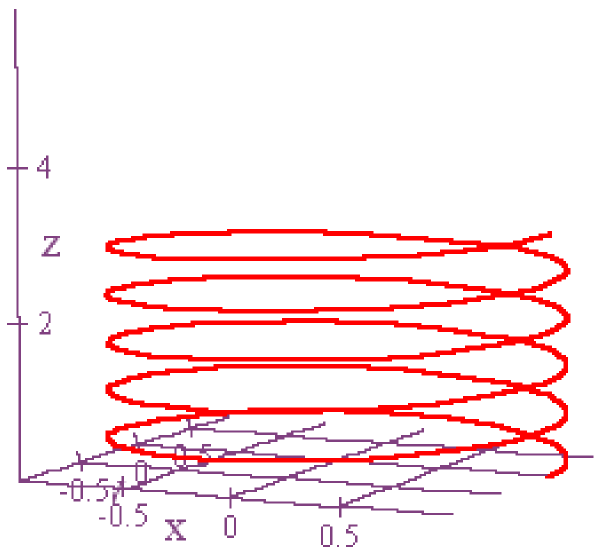
And if we want to elevate this circle to $z = 4$, all we have to do is change the fixed value of z to four.

$$\begin{aligned} &x = \cos(t) \\ &y = \sin(t) \\ &z = 4 \\ &0 \le t \le 2\pi \end{aligned}$$

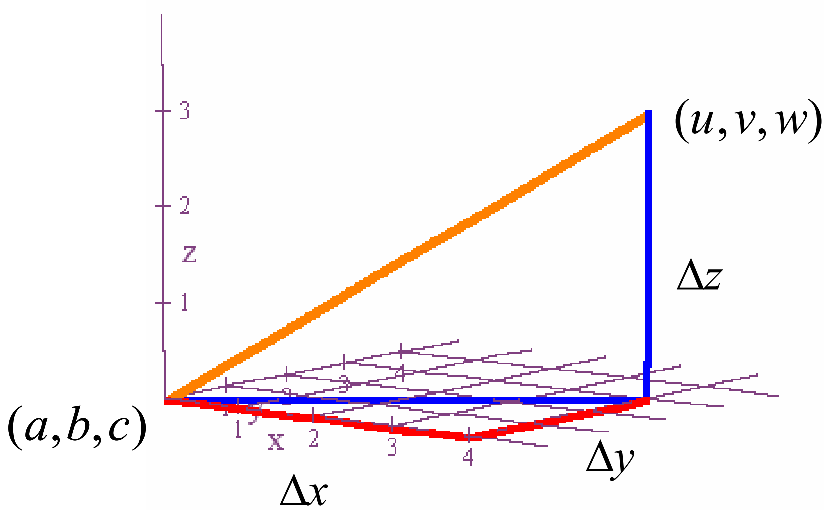


A fun variation we can do of a circle is a spiral that is technically known as a helix. However, I like to call it a slinky. To make a helix (or slinky), set your parametric equations so that x and y describe a circle, and then let z gradually increase as t increases.

$$x = \cos(t)$$
$$y = \sin(t)$$
$$z = t/10$$
$$0 \le t \le 10\pi$$



So that's it for circles! Now let's start looking at lines. Suppose you want to define a line segment parametrically that starts at (a,b,c) and ends at (u,v,w).



Then I claim that the parametric equations are as follows:

$$\begin{aligned} x &= a + t \cdot \Delta x \\ y &= b + t \cdot \Delta y \\ z &= c + t \cdot \Delta z \\ 0 &\le t \le 1 \end{aligned}$$

Certainly, when $t = 0$ we are at the point (a,b,c), and when $t = 1$ we add just the right amount of change to *x*, *y*, and *z* to take us to the point (u,v,w). Also, note the following two things. First, in our parametric equations we have *x*, *y*, and *z* all given as linear functions of *t*. Furthermore, by contemplating the diagram above, we should be able to convince ourselves that the object created by this parametrization will be a line. For example, when $t = \frac{1}{2}$, we will arrive at half our total change for *x*, *y*, and *z*, and the diagram suggests that we will have covered half the straight line distance between (a,b,c) and (u,v,w). Now let's do a particular example.

Example 1: Find parametric equations for the line segment from $(1,2,3)$ to $(4,7,5)$.

First, we write set *x*, *y*, and *z* equal to the coordinates of our starting point.

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

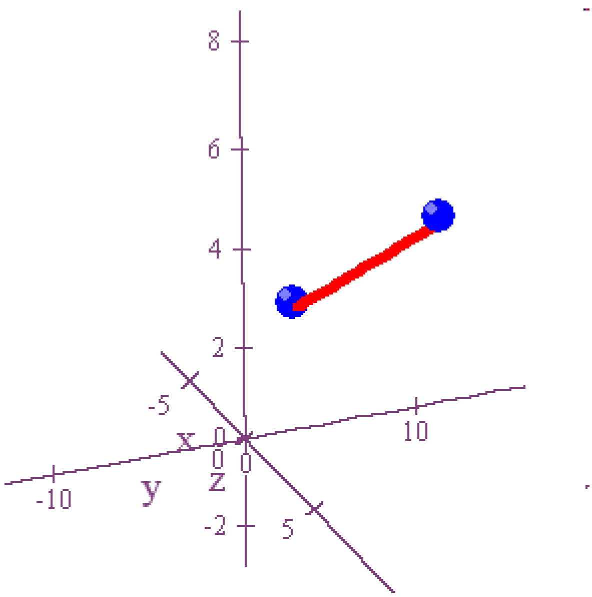
Next, we find the change associated with each variable.

$$\begin{aligned}\Delta x &= 4-1=3\\ \Delta y &= 7-2=5\\ \Delta z &= 5-3=2\end{aligned}$$

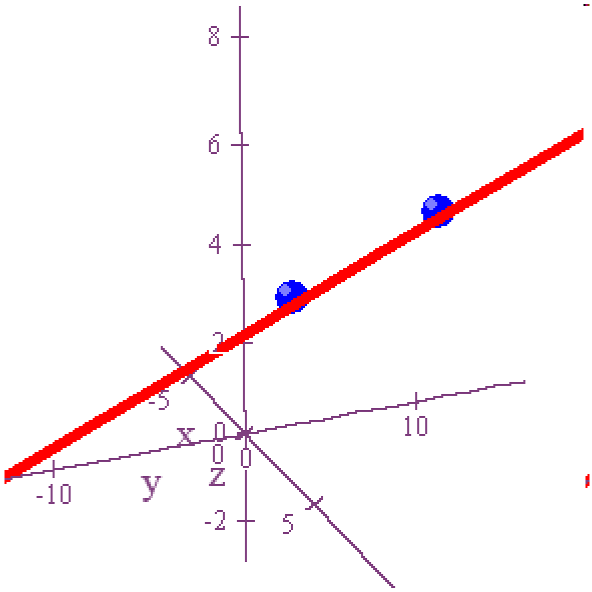
And finally, we multiply each change by t, add to our starting point, and let t vary from 0 to 1.

$$\begin{aligned}x &= 1+3t\\ y &= 2+5t\\ z &= 3+2t\\ 0 &\le t\le 1\end{aligned}$$

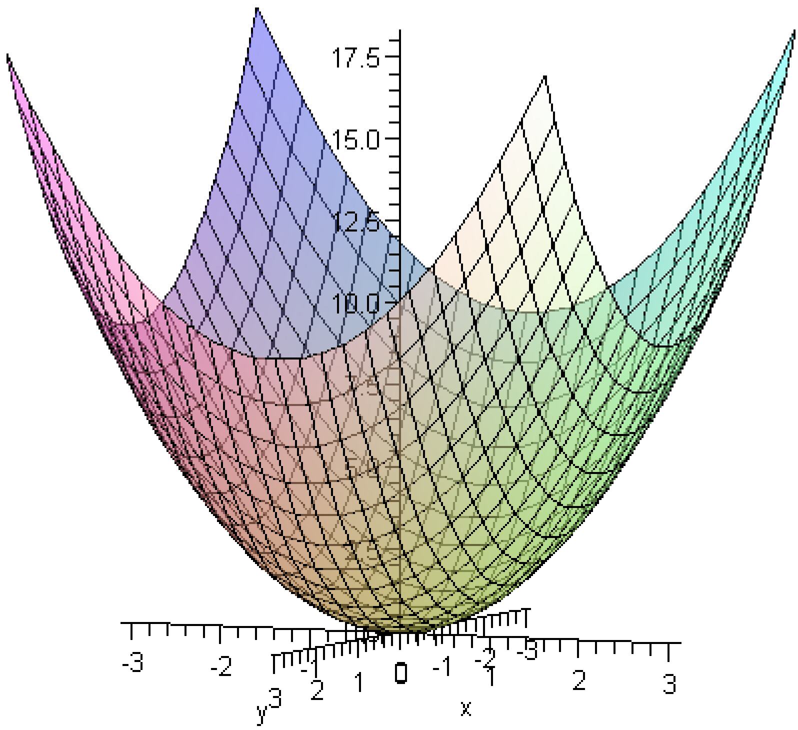
The end result is a very nice line segment from one point to another.



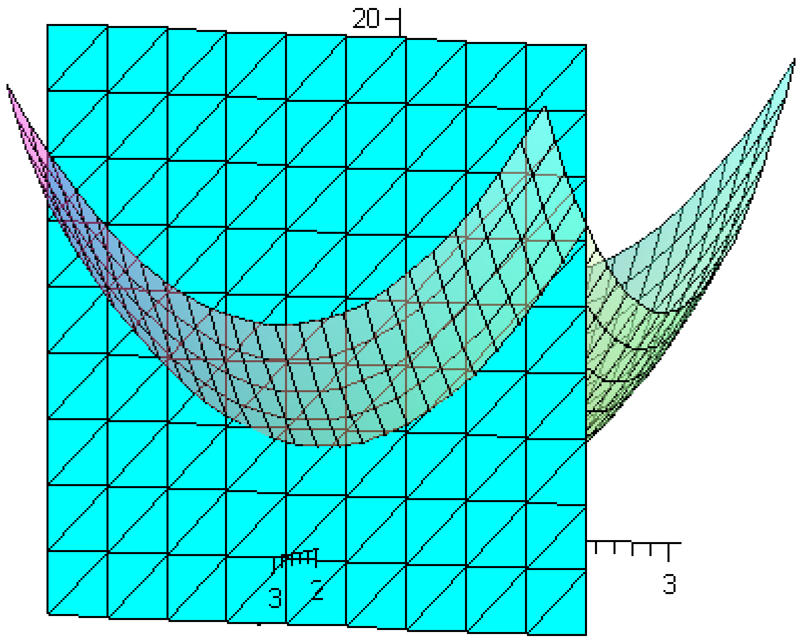
And if you want to extend the line, you just change the range for your parameter to $-\infty < t < \infty$.



And that's how you do a line! Now let's look at how to graph a curve of intersection along a surface. First, let's look at the graph of $z = x^2 + y^2$. That's our favorite paraboloid!

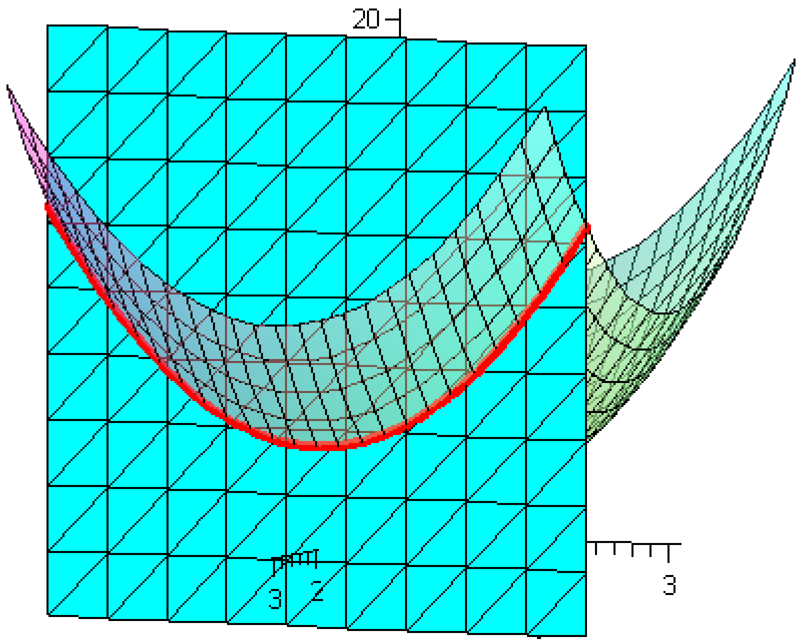


Now let's slice through this surface with the plane $x = 2$.

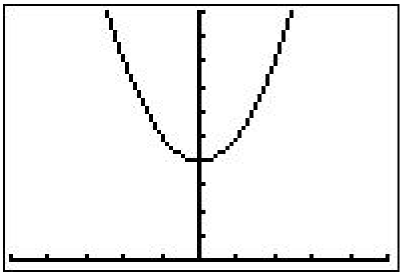


The result is a nice, parabolic cross-section. As we've seen before, we can get the equation for this cross-section by setting $x = 2$ in our formula for the parabolid. This gives us $z = 2^2 + y^2 = 4 + y^2$. To now create a parametrization for this curve in 3-dimensional space, one thing, at least, should be clear. We should fix x to the value 2. And now the rest of it is really quite simple. Jut set $y = t$ and then $z = 4 + t^2$. This is really just another way of saying that $z = 4 + y^2$. And finally, for the range of values of our parameter, for the graph above it will suffice to use $-3 \le t \le 3$ since that is the range I used in the graph for x and y. Okay, we're now ready to look at the parametric equations and the resulting graph.

$$x = 2$$
$$y = t$$
$$z = 4 + t^2$$
$$-3 \le t \le 3$$

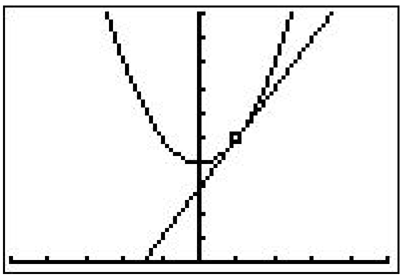


Success! However, we can do even more. Suppose we look at the graph of our cross-section $z = 4 + y^2$ in just 2-dimensions. Then we get something like the following.



If we set $y = 1$ on this graph, then $z = 4 + 1^2 = 5$, and we can use standard calculus techniques to find the slope of the tangent line to this curve at the point $(1,5)$.

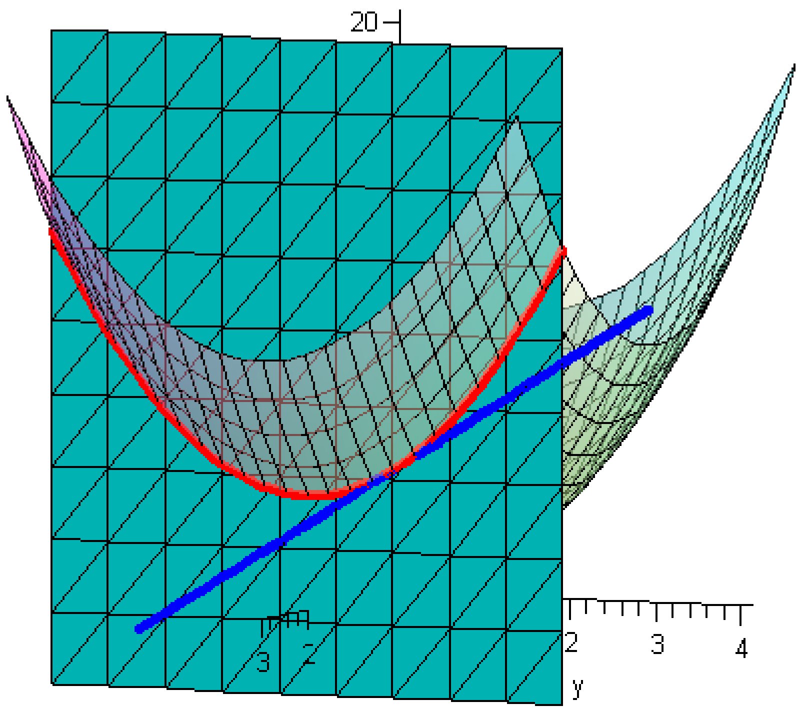
Clearly, $\frac{dz}{dy} = 2y$ and $\left.\frac{dz}{dy}\right|_{y=1} = 2$. Thus, an equation for our tangent to this 2-dimensional curve is $z = 2(y-1) + 5 = 2y + 3$.



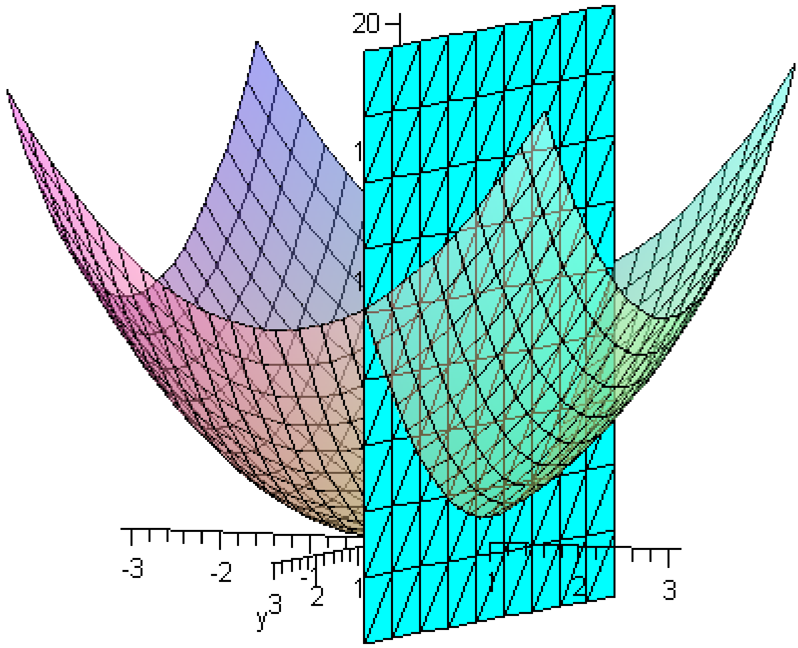
Now the question we want to ask ourselves is can we easily move all of this back onto our 3-dimensional surface, and the answer is yes, if we define our tangent line parametrically. First of all, since our point in 2-dimensions was $(y,z) = (1,5)$ and since we want everything to wind up in the plane $x = 2$, the point on the surface at which we want to place our tangent line is $(x,y,z) = (2,1,5)$. Second, since the slope of our tangent line in 2-dimensions was 2, in our parametric equations all we need to do is make sure that z grows twice as fast as y. All of this will happen perfectly if we set our parametric equations to:

$$\begin{aligned} &x = 2 \\ &y = 1 + t \\ &z = 5 + 2t \\ &-\infty < t < \infty \end{aligned}$$

Notice that these are parametric equations for a line that passes through $(2,1,5)$, the x-value is permanently fixed at 2 which will put our line in the plane $x = 2$, and z grows twice as fast as y. Now let's look at the result.

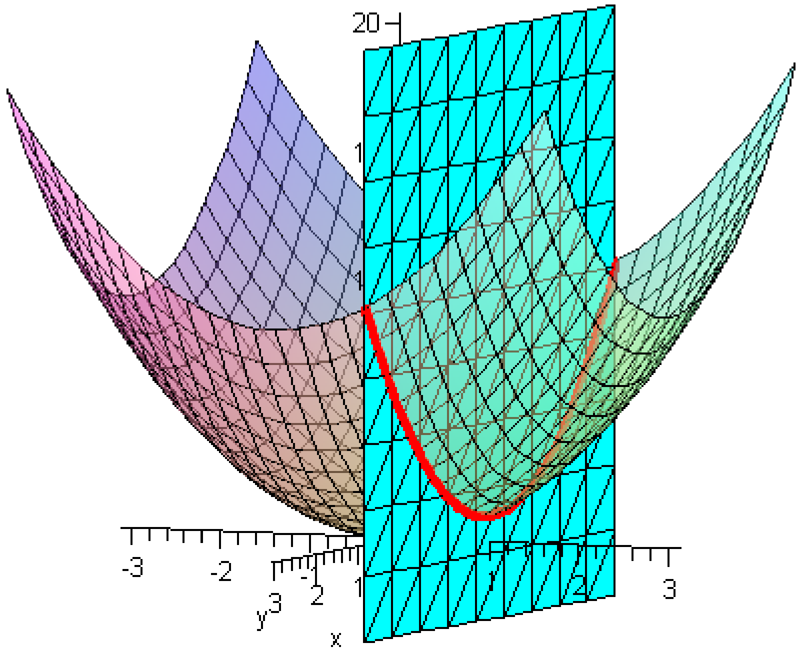


Perfect! Now let's do exactly the same thing, but this time we'll fix the y-value. Since the point at which we want to construct a tangent line to the surface is $(2,1,5)$, we'll intersect our surface with the plane $y=1$.



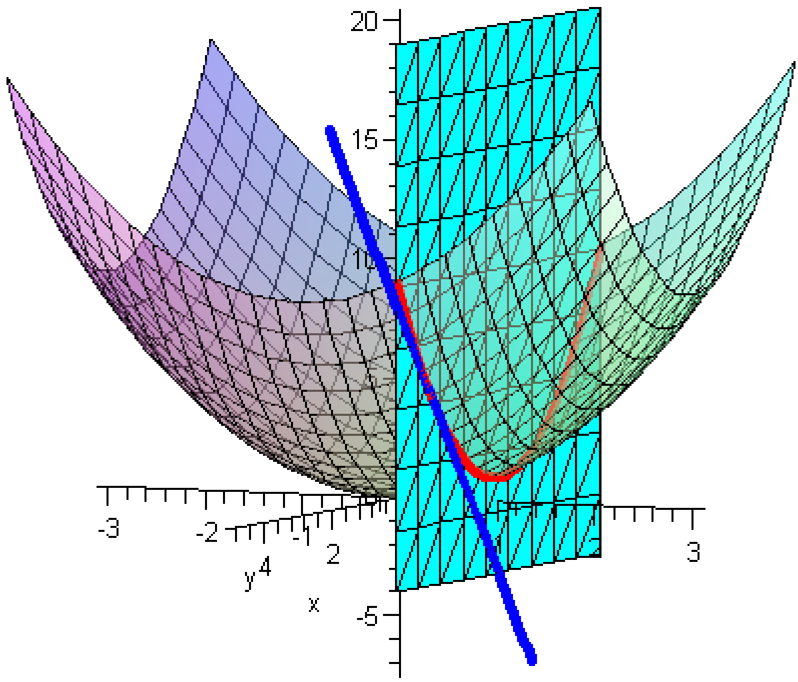
So far so good! Now let's add the graph of the cross-section to the surface. In this case, we'll have $z = x^2 + 1^2 = x^2 + 1$, and so our parametric equations should be:

$$
\begin{aligned}
&x = t \\
&y = 1 \\
&z = t^2 + 1 \\
&-3 < t < 3
\end{aligned}
$$

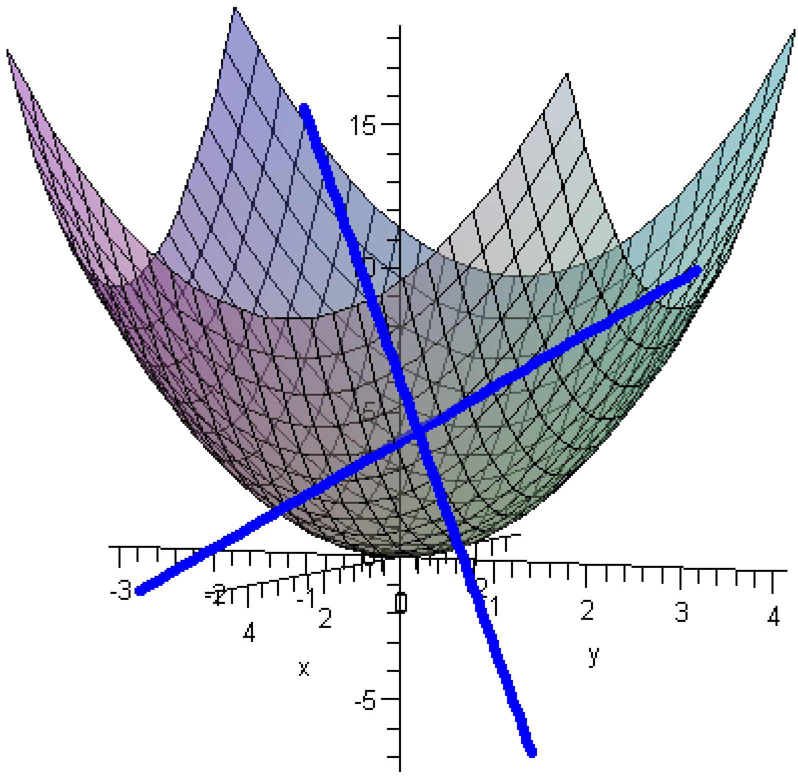


Most excellent! Now let's see if we can add the tangent line to the point $(2,1,5)$ that lies in the plane $y=1$. To get the slope of the tangent line, we'll consider $z = x^2 + 1$ as a function of one variable and differentiate to get $\frac{dz}{dx} = 2x$. If we evaluate this derivative at $x = 2$, we get that the slope of our tangent line is 4. We don't really need to construct the equation in 2-dimensions as we did previously since all we need is the slope in order to finish setting up parametric equations for the line in 3-dimensions. Just remember that, in this problem, if *x* increases by *t*, then *z* has to increase by 4*t* so that the slope of the line will be 4. And the parametric equations for this second tangent line at the point $(2,1,5)$ are:

$$x = 2 + t$$
$$y = 1$$
$$z = 5 + 4t$$
$$-\infty < t < \infty$$



Ah, yes, perfection is a beautiful thing! If we now, however, look at the graph of our original parabolid with only the two tangent lines at the point $(2,1,5)$, then we might notice something very important. Namely, these two lines define a unique plane that is tangent to our surface at the point $(2,1,5)$. As we proceed through this book this notion of a tangent plane will become increasingly important as it is simply the higher dimensional version of the tangent line that you undoubtedly studied in first semester calculus.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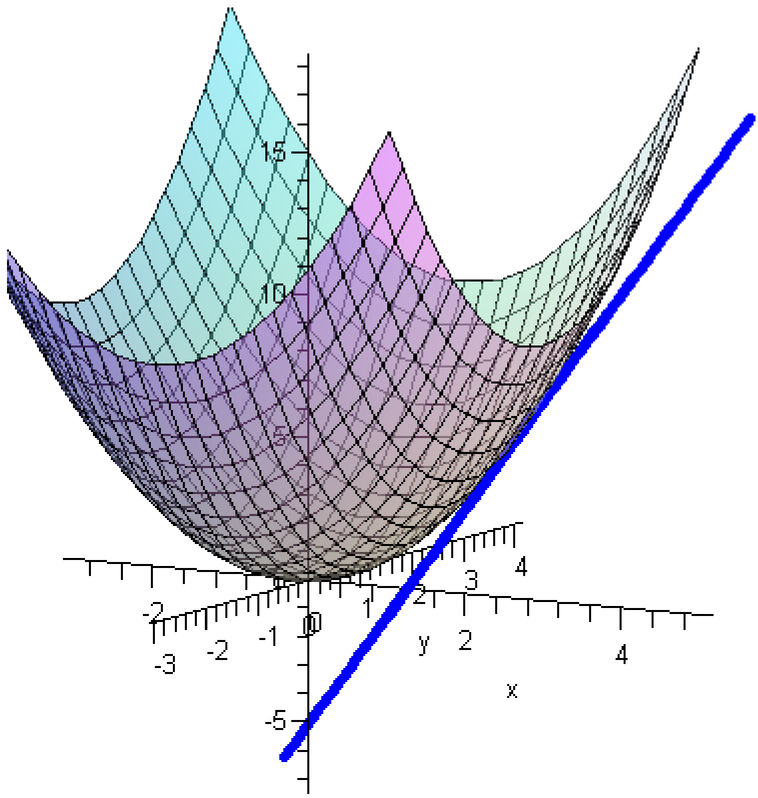
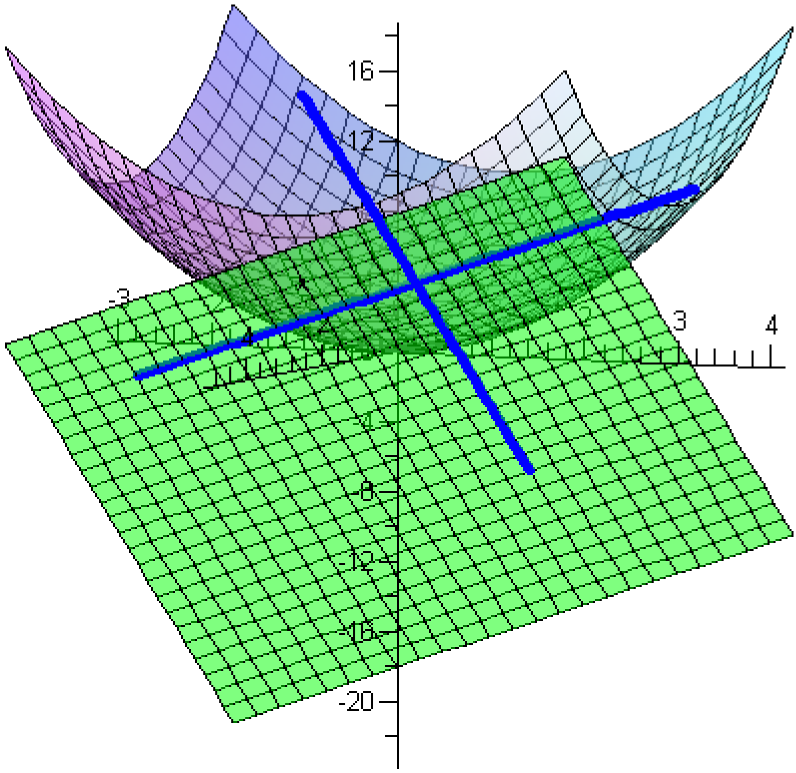


The best question now that we can ask ourselves is whether or not we know enough to find an equation for this tangent plane. Fortunately, the answer is yes! What we do know is that slope of the tangent line in the direction of the positive x-axis is 4, the slope of the tangent line in the direction of the positive y-axis is 2, and the point $(2,1,5)$ is in the tangent plane. Furthermore, we know that $z = Ax + By + C$ is an equation for a plane and that the coefficients of x and y correspond to slopes of tangent lines in directions of positive x and positive y, respectively. Thus, plugging in what we know, we get $5 = 4(2) + 2(1) + C \Rightarrow C = -5$. Hence, the equation for the tangent plane is $z = 4x + 2y - 5$.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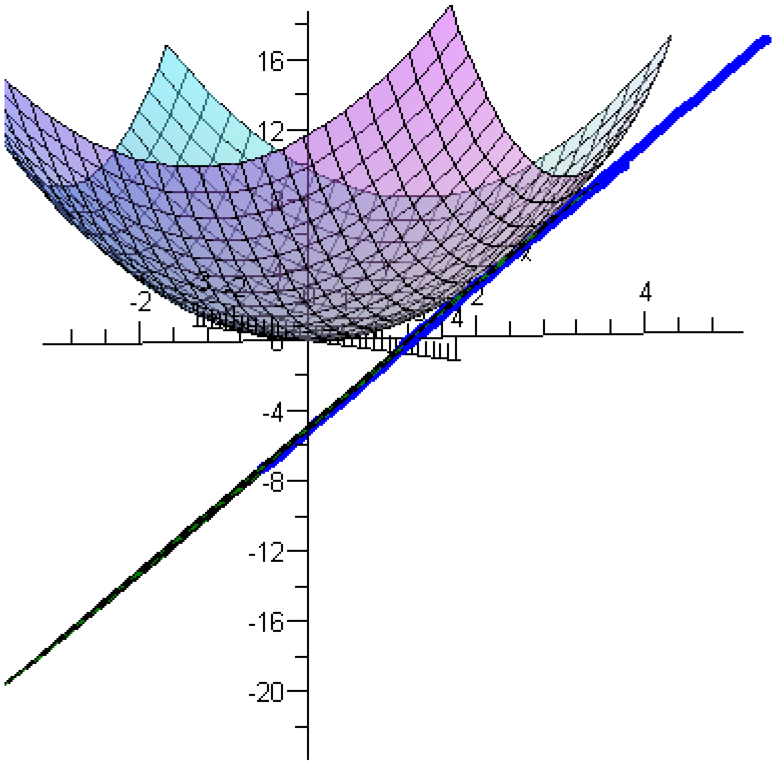

I love it when a construction comes together!