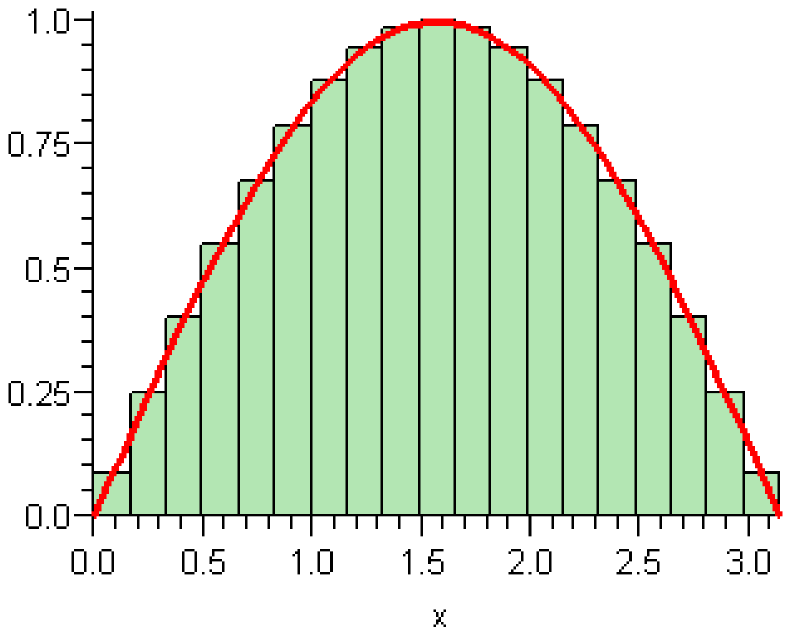
CHAPTER 8

INTEGRALS

You'll undoubtedly recall that you first learned about integrals by trying to figure out how to find the area under a curve.



The strategy was to subdivide our interval from a to b into a series of n subintervals of width $\Delta x = \dfrac{b-a}{n}$, evaluate our function at a point in each subinterval in order to get a height for a rectangle, add up the areas of the rectangles, and use that as an approximation for the area under the curve. Our expectation, of course was that as the number of subintervals increase, the approximation would get better and better. Thus, for the case in which we had $f(x) \geq 0$, we wrote

$$Area = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum f(x)\cdot \Delta x = \lim_{\Delta x\to 0} \sum f(x)\cdot \Delta x$$

And to help with computational matters, we soon discovered two versions of the Fundamental Theorem of Calculus.

<u>The Fundamental Theorem of Calculus:</u> Let $y = f(x)$ be continuous on the interval $a \leq x \leq b$. Then,

1. (The Derivative of the Integral) If $A(x) = \int_a^x f(u)\,du$, then $\dfrac{dA}{dx} = f(x)$.

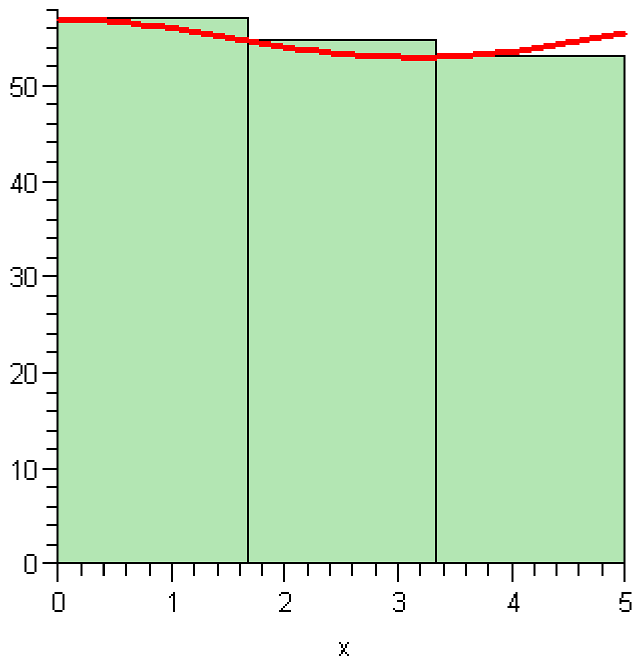
2. (The Integral of the Derivative) If $F(x)$ is any antiderivative of $f(x)$, then

   $\int_a^b f(x)\,dx = F(b) - F(a)$.

As a consequence of this theorem, evaluating integrals boiled down to an often simple process of finding antiderivatives of familiar functions, and much of single variable calculus is devoted to the abstract art of finding such antiderivatives. Nonetheless, one also encounters more applied problems such as this one.

Problem: Suppose you are driving down a highway, and your speed fluctuates in such a way that at time t hours, your speed is $f(t) = 55 + 2\cos(t)\dfrac{miles}{hour}$. Find the distance you have traveled after 5 hours.

The apparently tricky thing in this problem is that our speed is variable. In fact, over time your speed will drift from a low of 53 miles per hour to 57 miles per hour. However, in the diagram below, we try to approximate the area under the curves using rectangles.



What becomes apparent from these rectangles is that since we are using a constant speed for the height of each rectangle, that means that we can use our familiar $\text{distance} = \text{rate} \times \text{time}$ formula to get the distance traveled over each subinterval. Furthermore, this distance corresponds to the area of the related rectangle. And additionally, it's easy to see that if we increase the number of rectangles, then we get a better approximation for the actual distance traveled when our speed is variable. However, increasing the number of rectangles also results in a better approximation of the area under the curve. Consequently, we can conclude that when we have a variable speed given by a function $f(t)$ on an interval $a \le t \le b$, then the distance traveled is given by $\int_a^b f(t)\,dt$. There is, however, one more important thing we can learn from this example, and that is how we deal with units in most integrals related to the real world. If we write in the units associated with $f(t)$ and *t*, then we can express our integral as follows.

$$\lim_{\Delta t \to 0} \sum f(t) \frac{miles}{hour} \cdot \Delta t \, hours = \left(\lim_{\Delta t \to 0} \sum f(t) \cdot \Delta t \right) \frac{miles}{hour} hours = \int_a^b f(t)\,dt \ hours$$

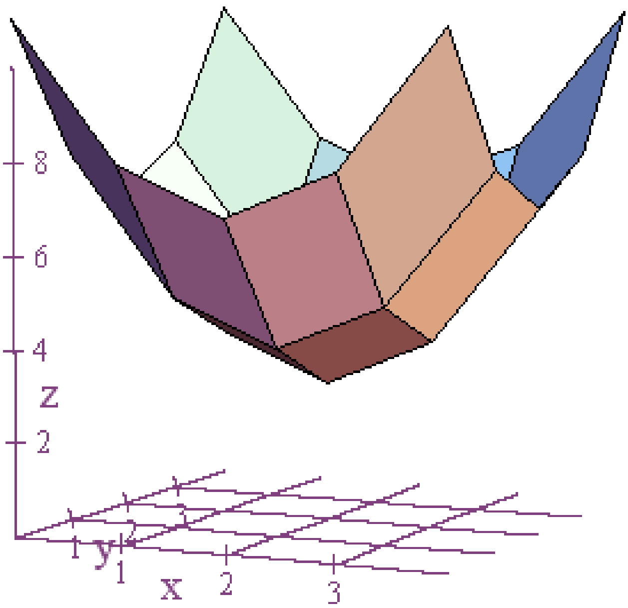
In other words, when units are involved, we can write the units on our integral as the product of the units on our function with the units on our input variable.

If we now have a function of two variables such as $z = f(x, y)$, we can define integrals in a similar way. In particular, suppose $z = f(x, y) = (x-2)^2 + (y-2)^2 + 3$,

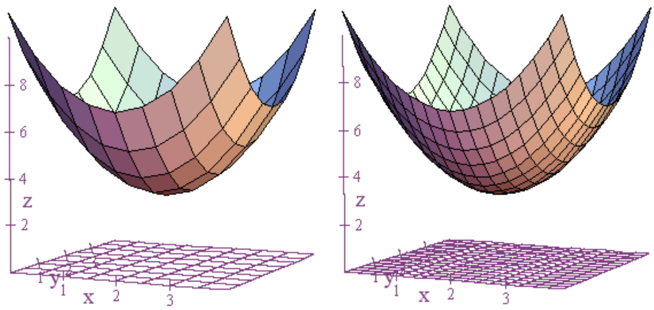
$0 \le x \le 4$, and $0 \le y \le 4$. Suppose also that the units of feet are attached to all three variables, *x*, *y*, and *z*. In this case, the expression $\sum f(x,y)\Delta x\Delta y$ will give us an approximation of the volume beneath our surface and above the *xy*-plane. Furthermore, if we multiply the output units by both of the input units, then we get units of cubic feet for our approximation. Additionally, in the above formula, since $\Delta x\Delta y$ is going to represent an element of area in the *xy*-plane, we often represent this area by ΔA. And finally, if we take the limit of the above expression as both Δx and Δy go to zero, then our volume approximation converges to the exact volume, and we call the result of such a limit a double integral since we have two input variables. We often write it like this,

$$\iint_R f(x,y)\,dA = \lim_{\Delta x,\Delta y \to 0} \sum f(x,y)\,\Delta A$$

In this formula, *R* is used to represent the region in the *xy*-plane that we are integrating over. The picture below uses $z = f(x,y) = (x-2)^2 + (y-2)^2 + 3$, $0 \le x \le 4$, and $0 \le y \le 4$, and it illustrates much of what we are talking about.



The grid in the *xy*-plane has the *x*-interval subdivided into four equal subintervals, and the *y*-interval is likewise subdivided into four subintervals. That gives a grid that contains sixteen rectangles total. Also, you see that if you plot the points $(x, y, f(x, y))$ above the corner points of each of our rectangles and if you then connected the dots, then you will get a graph that approximates the surface graph for $z = f(x, y) = (x-2)^2 + (y-2)^2 + 3$. Now suppose that for each of our sixteen rectangles we take the corner point with the smallest values for *x* and *y* and we use the corresponding *z*-value as the height of a box that we erect over our rectangle. Then the sum of the volume of the boxes will approximate the volume under our surface and above the *xy*-plane. Furthermore, as we take finer and finer subdivisions of both our *x*-interval and our *y*-interval, we get not only better approximations for the volume, but also a graph that better approximates our actual surface. Below is the grid and graph that results from subdividing each interval into eight parts, and this is followed by a picture where each side of the grid is subdivided into sixteen parts. As you can see, this improves the quality of the graph, and it will also improve any volume approximations.



The question we want to address now concerns how we go about actually evaluating $\iint_R f(x,y)\,dA = \lim_{\Delta x,\Delta y \to 0} \sum f(x,y)\,\Delta A$. The answer, fortunately, is given by a result known as Fubini's Theorem. When I was a sophomore in college taking multivariable calculus for the first time, my teacher described Fubini's Theorem as saying that if you are doing integration of functions of several variables and you get any answer at all, then it must be right! Well, Fubini's Theorem is not that generous, but close!

<u>Fubini's Theorem:</u> If $z = f(x,y)$ is continuous on the rectangle

$R = \{(x,y):\ a \le x \le b,\ c \le y \le d\}$, then $\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dydx = \int_c^d \int_a^b f(x,y)\,dxdy$.

These last two integrals are called iterated integrals, and it simply means that you first evaluate the inner integral, and then you take that result and integrate again. What Fubini's Theorem then tells us is that it doesn't matter whether you integrate first with respect to *x* or first with respect to *y*. The end result will be the same. Also, just as we treat other variables as constants when we differentiate with respect to *x*, we do the same when integrating with respect to *x*. In other words, whenever you integrate with respect to one particular variable, you always treat the other variables as constants. Now let's use $z = f(x,y) = (x-2)^2 + (y-2)^2 + 3$, $0 \le x \le 4$, and $0 \le y \le 4$ as an example. We'll find the volume under the surface in two ways, first by integrating with respect to *y* and then *x*, and then by doing the opposite order. Also, since both *x* and *y* vary from 0 to 4, we'll have the same limits of integration on both integrals both times. Here we go!

$$\iint_R f(x,y)\,dA = \int_0^4 \int_0^4 \left[(x-2)^2 + (y-2)^2 + 3\right] dydx = \int_0^4 \left[(x-2)^2 y + \frac{(y-2)^3}{3} + 3y \right]\Bigg|_0^4 dx$$

$$= \int_0^4 \left[(x-2)^2 4 + \frac{8}{3} + 12 \right] dx = \frac{4(x-2)^3}{3} + \frac{8x}{3} + 12x \Bigg|_0^4 = \frac{32}{3} + \frac{32}{3} + 48 = \frac{208}{3}$$

Similarly,

$$\iint_R f(x,y)\,dA = \int_0^4\int_0^4 \left[(x-2)^2+(y-2)^2+3\right]dxdy = \int_0^4 \left[\frac{(x-2)^3}{3}+(y-2)^2x+3x\right]\Bigg|_0^4 dx$$

$$= \int_0^4 \left[\frac{8}{3}+(y-2)^2 4+12\right]dy = \frac{8y}{3}+\frac{4(y-2)^3}{3}+12y\Bigg|_0^4 = \frac{32}{3}+\frac{32}{3}+48=\frac{208}{3}$$

We get the same result no matter what order we do our integration in, and that's exactly what Fubini's Theorem says ought to happen. Now let's look at a proof of sorts. It's not a full-blown, fully accurate proof. Instead, it's more of an outline of how the proof goes with the messier details omitted. Thus, I'll just call it an argument. The argument depends, though, on familiar properties of real numbers such as the commutative property of addition. In other words, when we are adding together the terms of the sum $\sum f(x,y)\Delta A = \sum f(x,y)\Delta y\Delta x$, we can add those terms up in any order we want. And as we take limits as $\Delta x, \Delta y \to 0$, this will result in different integral expressions that, nonetheless, give us the same overall result. Throughout I try to avoid cluttering up my expressions with subscripts, but I will use *i* to represent the i^{th} subinterval along the *x*-axis and *j* to represent the j^{th} subinterval along the *y*-axis. Here goes!

Argument: As any idiot can plainly see,

$$\iint_R f(x,y)\,dA \approx \sum_{i,j} f(x,y)\,\Delta A = \sum_i \left(\sum_j f(x,y)\,\Delta y \right) \Delta x = \sum_j \left(\sum_i f(x,y)\,\Delta x \right) \Delta y$$

Thus,

$$\iint_R f(x,y)\,dA = \lim_{\Delta x, \Delta y \to 0} \sum_{i,j} f(x,y)\,\Delta A = \lim_{\Delta x \to 0} \sum_i \left(\lim_{\Delta y \to 0} \sum_j f(x,y)\,\Delta y \right) \Delta x = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Similarly,

$$\iint_R f(x,y)\,dA = \lim_{\Delta x, \Delta y \to 0} \sum_{i,j} f(x,y)\,\Delta A = \lim_{\Delta y \to 0} \sum_j \left(\lim_{\Delta x \to 0} \sum_y f(x,y)\,\Delta x \right) \Delta y = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Consequently,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dydx = \int_c^d \int_a^b f(x,y)\,dxdy \qquad \blacksquare$$

And now, here's another example of Fubini's Theorem in practice.

$$\int_0^1 \int_2^4 xy^2\,dydx = \int_0^1 \left(\frac{xy^3}{3} \right)\Bigg|_2^4 dx = \int_0^1 \left(\frac{64x}{3} - \frac{8x}{3} \right) dx = \int_0^1 \frac{56x}{3}\,dx = \frac{28x^2}{3}\Bigg|_0^1 = \frac{28}{3} - \frac{0}{3} = \frac{28}{3}$$

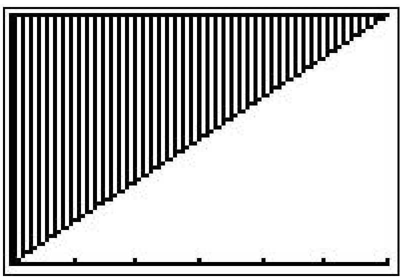
Likewise,

$$\int_2^4 \int_0^1 xy^2\,dxdy = \int_2^4 \left(\frac{x^2y^2}{2} \right)\Bigg|_0^1 dy = \int_2^4 \left(\frac{y^2}{2} - \frac{0}{2} \right) dy = \int_2^4 \frac{y^2}{2}\,dy = \frac{y^3}{6}\Bigg|_2^4 = \frac{64}{6} - \frac{8}{6} = \frac{56}{6} = \frac{28}{3}$$

Often times we change the order of integration simply because the integral is impossible for us to do by hand in one order, but very easy in the other. Here's one such example. We start with $\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\,dydx$. Well, I'm sorry, but I don't know an antiderivative, with respect to *y*, for $x\sqrt{y^3+1}$. Hence, let's see if we can rewrite this so that we integrate with respect to *x* first. If we do, however, then we'll generally also need to make some adjustments to our limits of integration. Now the limits we have on these integrals tell us that,

$$0 \le x \le 6 \text{ and}$$
$$\frac{x}{3} \le y \le 2$$

If we graph this region, we get something like the following.



What we need to do is express this same region in terms of inequalities such that this time *y* varies from one number to another and *x* varies from one function of *y* to another function of *y*. By doing this, we will determine our limits of integration for the new integral. Well, it's pretty clear (hopefully!) that we want $0 \le y \le 2$. Also, if we take our equation for the diagonal line, $y = \frac{x}{3}$, and solve it for *x*, we get $x = 3y$.

Therefore, our second inequality should be $0 \le x \le 3y$. Hence, the new integral is

$\int_0^2 \int_0^{3y} x\sqrt{y^3+1}\,dxdy$, and this one is pretty easy to do.

$$\int_0^2 \int_0^{3y} x\sqrt{y^3+1}\,dxdy = \int_0^2 \left. \frac{x^2\sqrt{y^3+1}}{2} \right|_0^{3y} dy = \int_0^2 \frac{9y^2\sqrt{y^3+1}}{2} dy = \int_1^9 \frac{3u^{1/2}}{2} du$$
$$= 9^{3/2} - 1^{3/2} = 27 - 1 = 26$$

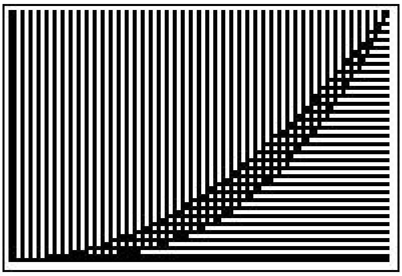
Another way we can use double integrals is to find areas. To do this, just think of your integrand as being $f(x, y) = 1$. In other words, if you have a solid whose base is some region *R*, but the height of that solid is 1, then the volume will numerically be the same as the area of the base. That's what we're doing here!

Example 1: Use a double integral to find the area of the region between the curves $y = x^2$ and $y = x^3$ from $x = 0$ to $x = 1$.

Solution: On this interval, the value of x^2 will be greater than that of x^3. Thus, our function and intervals for *x* and *y* are,

$$z = f(x, y) = 1$$
$$0 \le x \le 1$$
$$x^3 \le y \le x^2$$

Also, here is a picture of the region *R* whose area we want to find,



Now all we need to do is evaluate the integral.

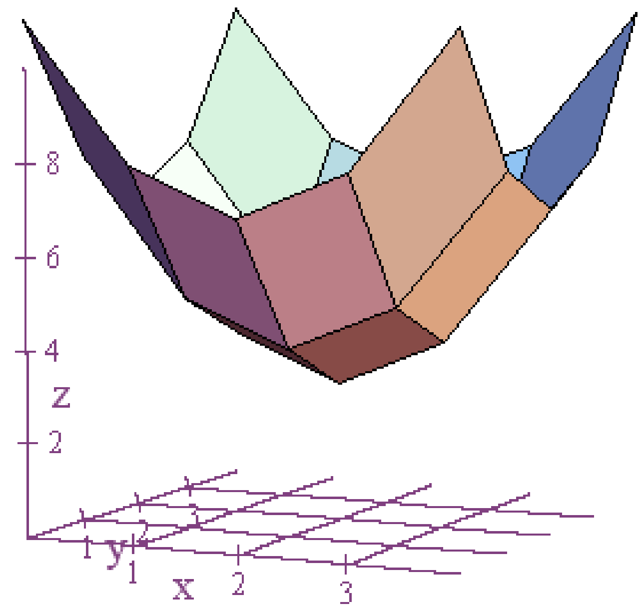
$$Area = \iint_R dA = \int_0^1 \int_{x^3}^{x^2} dydx = \int_0^1 y \Big|_{x^3}^{x^2} dx = \int_0^1 \left(x^3 - x^2\right) dx = \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_0^1$$
$$= \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{0}{3} - \frac{0}{4}\right) = \frac{4}{12} - \frac{3}{12} = \frac{1}{12}$$

We can also reverse the order of integration here and integrate first with respect to x. If we do that, then we have to find a numerical interval over which y varies and a pair of functions of y that x will lie between. In this case, it looks like we want $0 \le y \le 1$ and $\sqrt{y} \le x \le \sqrt[3]{y}$. Thus,

$$Area = \iint_R dA = \int_0^1 \int_{\sqrt{y}}^{\sqrt[3]{y}} dxdy = \int_0^1 x \Big|_{\sqrt{y}}^{\sqrt[3]{y}} dy = \int_0^1 \left(y^{1/3} - y^{1/2}\right) dy = \left(\frac{3y^{4/3}}{4} - \frac{2y^{3/2}}{3}\right)\Bigg|_0^1$$
$$= \left(\frac{3}{4} - \frac{2}{3}\right) - \left(\frac{0}{4} - \frac{0}{3}\right) = \frac{9}{12} - \frac{8}{12} = \frac{1}{12}$$

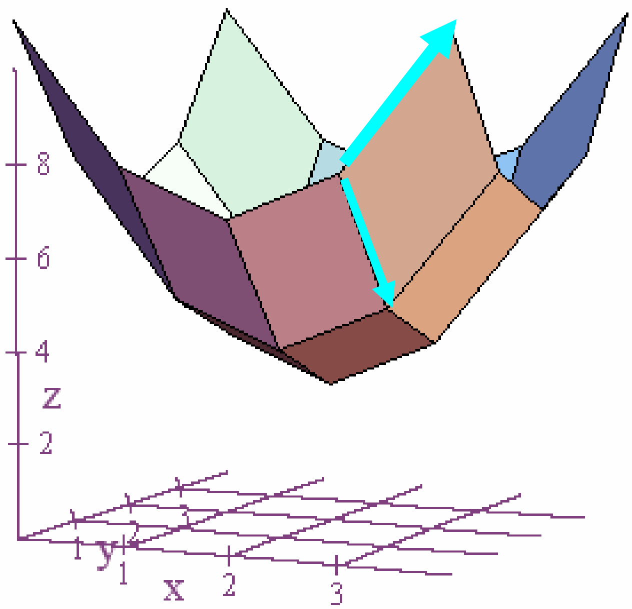
And we get the same answer! Exactly as Fubini's Theorem predicts.

Another application of double integrals is to compute surface area. For example, let's take a function we looked at earlier, $z = f(x,y) = (x-2)^2 + (y-2)^2 + 3$ with $0 \le x \le 4$ and $0 \le y \le 4$. If we look closely at the wireframe graph below, we might notice that above each rectangle in our region *R* in the *xy*-plane there lies a corresponding parallelogram whose area approximates the surface area above that rectangle.

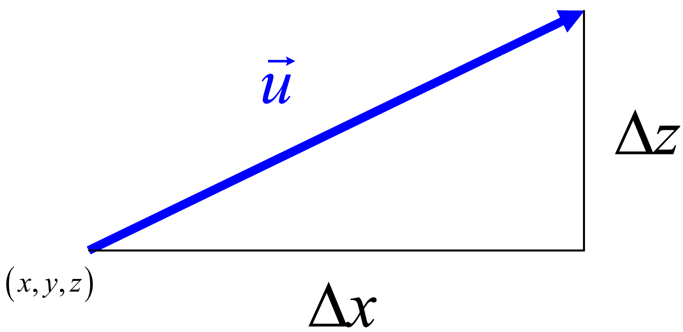


If we find the areas of these parallelograms and add them up, then the sum will approximate the area of our surface. Furthermore, as we partition our region *R* into smaller and smaller subrectangles, our approximation of the surface area should become more accurate. The question now is how do we find the areas of our parallelograms? And the answer is, vectors! Recall that if a have a parallelogram defined by two vectors $\vec{u}$ and $\vec{v}$, then the area of that parallelogram is given by

$\|\vec{u}\times\vec{v}\|$. Below is our wireframe drawing again, but this time with two vectors added to one of the parallelograms on the surface.



If we designate, on our rectangle in the *xy*-plane, the corner point with the smallest coordinates as (x, y), then the two adjacent corner points will have coordinates $(x+\Delta x, y)$ and $(x, y+\Delta y)$. Thus, I claim we can define our vectors $\vec{u}$ and $\vec{v}$ as $\vec{u} \approx \Delta x\,\hat{i} + 0\,\hat{j} + \dfrac{\partial f}{\partial x}\Delta x\,\hat{k}$ and $\vec{v} \approx 0\,\hat{i} + \Delta y\,\hat{j} + \dfrac{\partial f}{\partial y}\Delta y\,\hat{k}$. Now the question is how do we get this? Well, let's answer for the vector $\vec{u}$ by looking at a typical diagram below.



In this diagram, our vector $\vec{u}$ lies in a plane parallel to the *xz*-plane, and hence, as we traverse the length of $\vec{u}$ there is a change in *x* and a change in *z*, but no change in *y*. Also, notice that $\vec{u} = \Delta x\,\hat{i} + \Delta z\,\hat{k}$ and that the slope of $\vec{u}$ is $\frac{\Delta z}{\Delta x}$. Realize, too, that if Δx is small, then $\vec{u}$ will approximate a tangent vector at (x,y,z) pointing in the direction of the positive *x*-axis, and thus, it's slope is also approximately equal to $\frac{\partial f}{\partial x}$ evaluated at (x,y,z). Hence, we get that $\frac{\Delta z}{\Delta x} \approx \frac{\partial f}{\partial x} \Rightarrow \Delta z \approx \frac{\partial f}{\partial x}\Delta x$, and $\vec{u} \approx \Delta x\,\hat{i} + 0\,\hat{j} + \frac{\partial f}{\partial x}\Delta x\,\hat{k}$. A similar argument with respect to *y* shows that $\vec{v} \approx 0\,\hat{i} + \Delta y\,\hat{j} + \frac{\partial f}{\partial y}\Delta y\,\hat{k}$, and both of these approximations improve as $\Delta x, \Delta y \to 0$. Consequently,

$$\vec{u} \times \vec{v} \approx \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & \frac{\partial f}{\partial x}\Delta x \\ 0 & \Delta y & \frac{\partial f}{\partial y}\Delta y \end{vmatrix} = \left(-\frac{\partial f}{\partial x}\Delta x\Delta y\right)\hat{i} - \left(\frac{\partial f}{\partial y}\Delta x\Delta y\right)\hat{j} + \left(\Delta x\Delta y\right)\hat{k}$$

And,

$$\|\vec{u}\times\vec{v}\| \approx \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \Delta x^2 \Delta y^2 + \left(\frac{\partial f}{\partial y}\right)^2 \Delta x^2 \Delta y^2 + \Delta x^2 \Delta y^2} = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \cdot \Delta x \Delta y$$

Therefore,

$$Surface\ Area = \lim_{\Delta x, \Delta y \to 0} \sum_{i,j} \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \cdot \Delta x \Delta y$$

$$= \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dxdy = \iint_R \left(\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \right) dA$$

Also, if we let S denote the surface we are integrating over, and if we denote an element of area on the surface by ΔS, then what we've also shown above is that

$\Delta S \approx \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \cdot \Delta A$, and hence,

$$Surface\ Area = \iint_S dS = \iint_R \left(\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \right) dA$$

Consequently, we're really turning a double integral over a surface S into a more manageable double integral over a region R in the plane. Additionally, remember the formula $dS = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA$ because we'll see it again in the chapter on Green's Theorem, Stokes' Theorem, and the Divergence TheoremI.

Finally, we have a very nice formula for surface area! Unfortunately, in practice this often leads to something that is hard to integrate, and that will be the case with $z = f(x,y) = (x-2)^2 + (y-2)^2 + 3$ with $0 \le x \le 4$ and $0 \le y \le 4$. In this instance,

$$\begin{aligned}
z_x &= 2(x-2) = 2x-4 \\
z_y &= 2(y-2) = 2y-4 \\
{z_x}^2 &= 4x^2 - 16x + 16 \\
{z_y}^2 &= 4y^2 - 16y + 16
\end{aligned}$$

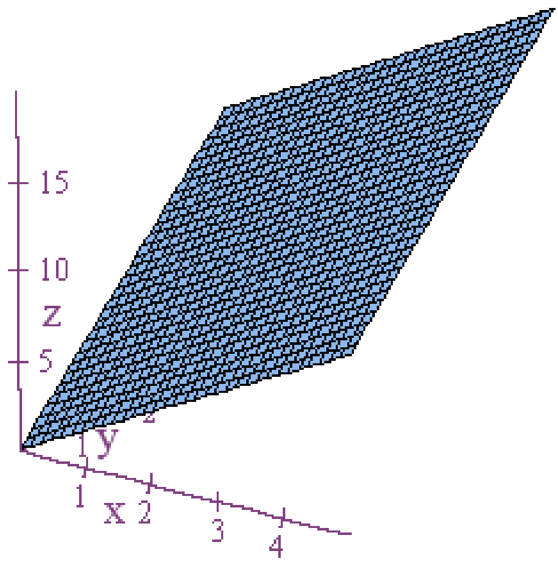
Hence,

$$\textit{Surface Area} = \iint_R \sqrt{4x^2 + 4y^2 - 16x - 16y + 33}\, dA$$

Have fun doing that one! Often times with integrals such as these we have to evaluate them numerically. However, later in this chapter we'll see how to do a similar integral by making a change of variables. In fact, the only reason we're not going to do it at this point is because our region *R* doesn't have the appropriate shape for the kind of change we have in mind. For now, though, take it on faith that if I plug the above integral into *MAPLE* software and ask for a numerical approximation of the answer, then I get back that the surface area is approximately 52 square units. Close enough for government work!

We should probably do at least one example that does lead to something we can integrate, so let's try this one.

Example 2: Find the surface area of the plane defined by $z = 2x + 3y$ where $0 \le x \le 5$ and $0 \le y \le 4$.



Solution: The necessary partial derivative computations are,

$$z_x = 2 \Rightarrow z_x^{\ 2} = 4$$
$$z_y = 3 \Rightarrow z_y^{\ 2} = 9$$

Thus,

$$Surface\ Area = \int_0^5 \int_0^4 \sqrt{14}\, dydx = \int_0^5 y\sqrt{14}\Big|_0^4 dx = \int_0^5 4\sqrt{14}\, dx = 4x\sqrt{14}\Big|_0^5 = 20\sqrt{14} \approx 74.833$$

We can verify this result by taking the vectors $\vec{u}$ and $\vec{v}$ that define this parallelogram portion of the plane, and then calculate $\|\vec{u} \times \vec{v}\|$. Thus, $\vec{u} = 5\hat{i} + 10\hat{k}$ and $\vec{v} = 4\hat{j} + 12\hat{k}$.

Hence,

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 0 & 10 \\ 0 & 4 & 12 \end{vmatrix} = -40\hat{i} - 60\hat{j} + 20\hat{k}$$

And,

$$\|\vec{u}\times\vec{v}\| = \sqrt{1600+3600+400} = \sqrt{5600} = \sqrt{400\cdot 14} = 20\sqrt{14} \approx 74.833$$

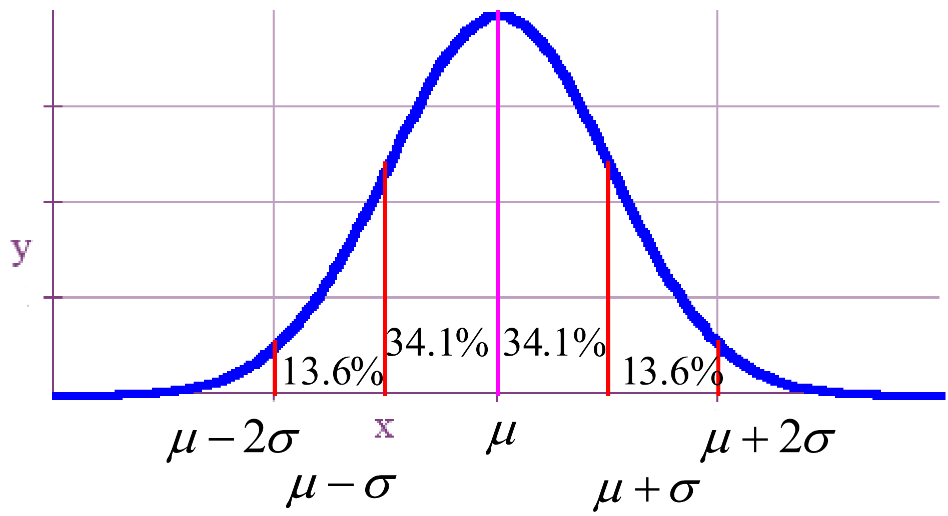
I love it when things work out!

We now want to look at a very different application of double integrals, but first we need to define what we mean by a probability density function (also called a probability distribution function). If we are dealing with a function of one variable, $p = p(x)$, then this function could be a probability density function if the following two conditions are met,

1. $p(x) \geq 0$ for all x

2. $\int_{-\infty}^{\infty} p(x)\,dx = 1$

The most widely known example of a probability density function of one variable is the one that gives rise to the bell-shaped curve known as the normal curve or normal distribution. It's defined by $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ where μ is the mean or average value of the distribution and σ is the standard deviation, a measure of how spread out the values of the distribution are. A typical normal curve looks like the following.



Also, as this picture suggests, we find probabilities in a probability density function by calculating the area under the curve. Thus, for example, in the normal distribution above there is a 34.1% chance that a score will fall between the mean and one standard deviation above the mean, and there is a 13.6% chance a score will be between $\mu + \sigma$ and $\mu + 2\sigma$.

When two variables are involved, a probability density function is also called a joint density function, and the two conditions a function $p(x, y)$ must follow in order to be a joint density function are,

1. $p(x,y) \geq 0$ for all x and y

2. $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} p(x,y)\,dydx = 1$

Given that we have a joint density function, to find the probability that $a \leq x \leq b$ and $c \leq y \leq d$ m we evaluate $P(a \leq x \leq b,\, c \leq y \leq d) = \int\limits_{a}^{b}\int\limits_{c}^{d} p(x,y)\,dydx$. Here's an easy example.

Example 3: Given the joint density function $p(x,y) = \begin{cases} x+y & \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \textit{otherwise} \end{cases}$, find the probability that $0 \leq x \leq \frac{1}{2}$ and $0 \leq y \leq \frac{1}{2}$.

Solution: Probability = $P\begin{pmatrix} 0 \leq x \leq \frac{1}{2} \\ 0 \leq y \leq \frac{1}{2} \end{pmatrix} = \int\limits_{0}^{1/2}\int\limits_{0}^{1/2} (x+y)\,dydx = \int\limits_{0}^{1/2} \left(xy + \frac{y^2}{2} \right)\Bigg|_0^{1/2} dx$

$= \int\limits_{0}^{1/2} \left(\frac{x}{2} + \frac{1}{8} \right) dx = \left(\frac{x}{2} + \frac{1}{8} \right)\Bigg|_0^{1/2} = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$. ■

Next, I want to show you a very useful theorem. It basically says that if you have two probability density functions of one variable, then it's very easy to construct a joint density function from them. For example,, both height and weight for adult men tend to be normally distributed. Using the theorem we will prove below, we can easily construct a corresponding joint density function for answering questions such as what is the probability that an adult male has a height between five feet and six feet and a weight between 180 pounds and 200 pounds?

<u>Theorem:</u> If $p(x)$ and $q(y)$ are both probability density functions of one variable, then $f(x, y) = p(x)q(y)$ is a joint density function.

<u>Proof:</u> It suffices to show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x)q(y)\,dydx = 1$. Clearly, though, since functions of one variable may be treated as constants when integrated with respect to another variable, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x)q(y)\,dydx = \int_{-\infty}^{\infty} p(x)\left(\int_{-\infty}^{\infty} q(y)\,dy\right)dx = \left(\int_{-\infty}^{\infty} q(y)\,dy\right)\left(\int_{-\infty}^{\infty} p(x)\,dx\right) = 1\cdot 1 = 1. \quad \blacksquare$$

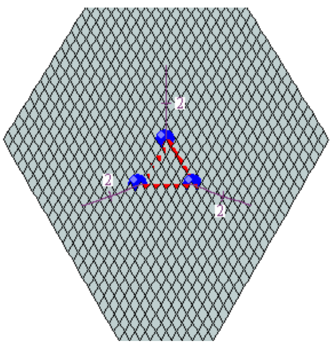
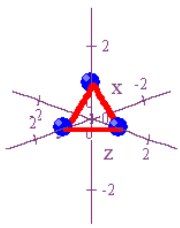
This proof also shows us how to evaluate a joint density function that has been constructed from two probability density functions of a single variable. We merely multiply two individually calculated probabilities together. For example, let's suppose that adult men have an average height of 5.75 feet (5 feet, 9 inches) with a standard deviation of 3 inches (0.25 feet), and that the average weight of an adult male is 190 pounds with a standard deviation of 10 pounds. Then, technically speaking, the probability that an adult male has a height between 5 feet and 6 feet is $\int_5^6 \frac{1}{0.25\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-5.75}{0.25}\right)^2} dx$, and the probability that an adult male has a weight between 180 pounds and 200 pounds is $\int_{180}^{200} \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-190}{10}\right)^2} dy$. Consequently, the joint probability that an adult male has a height between 5 feet and 6 feet and a weight between 180 pounds and 200 pounds is

$\int_5^6 \frac{1}{0.25\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-5.75}{0.25}\right)^2} dx \cdot \int_{180}^{200} \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-190}{10}\right)^2} dy$. For those who know a little probability theory, this product should come as no surprise because whenever we are finding the probability of two events joined by the word "and," we generally multiply simpler probabilities together. In this case, though, we can't find simple expressions for the antiderivatives of our integrands, and so we have to evaluate these integrals numerically. Fortunately, these days even a TI-83 or TI-84 calculator can easily do the math. Using the normal cumulative distribution key on this calculator, we get that the probability that the height is from 5 to 6 feet is approximately 0.84, and the probability that the weight is in the specified range is about 0.68. Thus, the probability that both these events occur is $0.84 \cdot 0.68 \approx 0.57$.

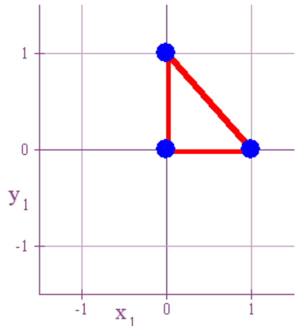
Now let's increase the level of complexity one notch above that of double integrals. Let's talk about triple integrals. Mathematically speaking, a triple integral is just an integral that is done with respect to a volume in three dimensions as opposed to an area in two dimensions that we integrate over when doing double integrals. Thus, in a triple integral, an element of volume is defined by a product of the change in *y* times the change in *x* times the change in *z*. Symbolically, we can write this as change in volume $= \Delta V = \Delta x \Delta y \Delta z$, and in differential form we write this as $dV = dxdydz$. Now let's suppose that we have a function of three variables such as $w = f(x, y, z)$ and that we want to integrate this function over a solid region *V* in three dimensional space. Then we write this integral as $\iiint_V f(x, y, z)\, dV$. Now we have to ask, how do we evaluate this integral? Fortunately, Fubini's Theorem can be proven for this higher dimensional case, and thus, we can write the integral as three separate iterated integrals. In particular, suppose the region *V* can be described by the inequalities $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, and $h_1(x, y) \le z \le h_2(x, y)$. Then $\iiint_V f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\, dzdydx$. By Fubini's Theorem, this triple integral can be written as an iterated integral in six different ways corresponding to the six different permutations we can make of the variables *x*, *y*, and *z*. Of course, some orders may lead to an easy integration while others lead to expressions that it's difficult or impossible to find simple antiderivatives for. Furthermore, if our integrand is equal to 1, then the result of our triple integral is volume $= \iiint_V dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} dzdydx$. Now let's look at a couple of example problems involving triple integrals.

<u>Example 4:</u> Find the volume of the solid in the first octant that is bounded below by the *xy*-plane and above by the plane $x + y + z = 1$.

<u>Solution:</u> The whole trick to triple integrals is figuring out how to describe your region of integration in terms of intervals involving the variables *x*, *y*, and *z*. A picture can be very helpful, but it still takes a certain amount of practice and ingenuity.

In this case, though, our solid is bounded above by the plane $x + y + z = 1$ and below by the *xy*-plane. This suggests that if we rewrite $x + y + z = 1$ as $z = 1 - x - y$, then we can say throughout that *z* varies from $z = 0$ to $z = 1 - x - y$. Consequently, all we need to do now is to describe the bottom part of our solid in terms of *x* and *y*. In two dimensions, our bottom part looks like this.



The corner points are at $(0,0)$, $(0,1)$, and $(1,0)$. Additionally, the line through $(0,1)$ and $(1,0)$ can be described by $y=-x+1$. Thus, we can describe the enclosed triangular region by the inequalities $0\le x\le 1$ and $0\le y\le -x+1$. We're now ready to list all our inequalities together and to find the volume of our solid.

$$\begin{aligned}&0\le x\le 1\\&0\le y\le -x+1\\&0\le z\le 1-x-y\end{aligned}$$

$$\text{Volume } = \iiint_V dV = \int_0^1\int_0^{-x+1}\int_0^{1-x-y} dzdydx = \int_0^1\int_0^{-x+1}(1-x-y)\,dydx$$

$$= \int_0^1 \left(y - xy - \frac{y^2}{2}\right)\Bigg|_0^{-x+1} dx = \int_0^1\left(-x+1-x(-x+1)-\frac{(-x+1)^2}{2}\right)dx$$

$$= \int_0^1\left(-x+1+x^2-x-\frac{x^2-2x+1}{2}\right)dx = \int_0^1\left(\frac{x^2}{2}-x+\frac{1}{2}\right)dx$$

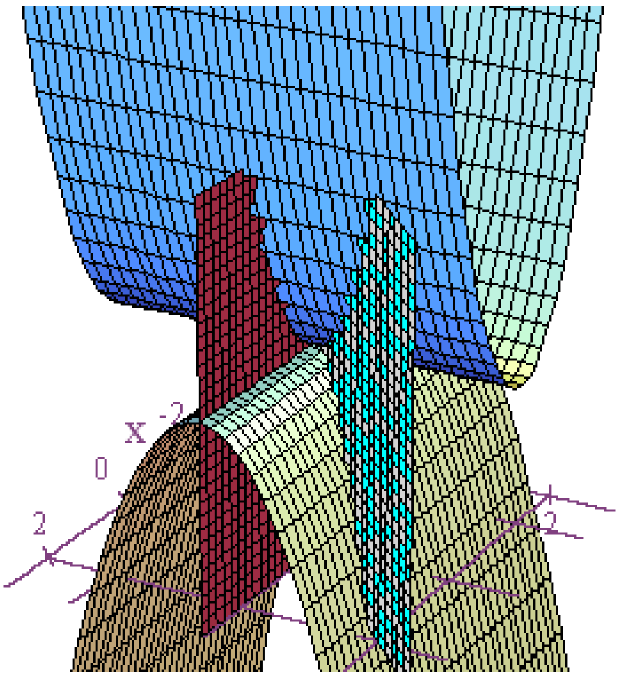
$$= \left(\frac{x^3}{6}-\frac{x^2}{2}+\frac{x}{2}\right)\Bigg|_0^1 = \frac{1}{6}$$

Any questions?

Our last example on triple integrals involves integrating a function over a solid volume.

Example 5: Let V be the solid region between the graphs of $z=-y^2$ and $z=x^2$ where $0 \le x \le 1$ and $0 \le y \le x$. Evaluate $\iiint_V (x+1)\,dV$.

Solution: The solid bounded by a variety of surfaces is not that easy to draw, but here's part of it.



Fortunately, it's still pretty easy to see how to describe our solid region in terms of inequalities.

$$0 \le x \le 1$$
$$0 \le y \le x$$
$$-y^2 \le z \le x^2$$

Thus,

$$\text{Volume } = \iiint_V dV = \int_0^1 \int_0^x \int_{-y^2}^{x^2} dzdydx = \int_0^1 \int_0^x \left(x^2 + y^2\right) dydx$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3} \right)\Bigg|_0^x dx = \int_0^1 \frac{4x^3}{3} dx = \frac{x^4}{3}\Bigg|_0^1 = \frac{1}{3}$$

Pretty simple!

Quite often what is a difficult integral in *xyz*-coordinates can be a very simple integral in other coordinate systems. At this point, the only other coordinate systems we are familiar with are polar, cylindrical, and spherical coordinates. We'll begin with double integrals in polar coordinates. It's often advantageous to switch to this coordinate system if the region R we are integrating over is circular or otherwise easy to describe in the polar coordinate system. Also, recall the following important relationships between polar coordinates and x and y.

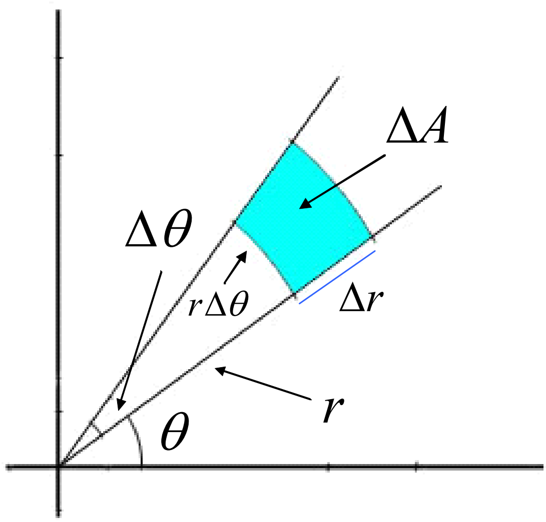
$$x = r \cdot \cos\theta$$
$$y = r \cdot \sin\theta$$
$$x^2 + y^2 = r^2$$

Recall also that when dealing with rectangular coordinates, we were able to write a double integral as an iterated integral with respect to y and x in the way we did because an element of area in our region R was equal to the product of a change in y times a change in x.

$$\Delta A = \Delta y \Delta x \Rightarrow dA = dydx$$

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dydx$$

Consequently, the question is if we are going to write $\iint_R f(x,y)\,dA$ as an iterated integral with respect to r and θ, then how do we express an element of area in terms of these input variables? The diagram below gives us the answer.



$$\Delta A \approx r\Delta\theta \cdot \Delta r = r\Delta r\Delta\theta$$
$$dA = rdrd\theta$$

In polar coordinates, our elements of area are portions of "pizza slices."

Consequently, they aren't the usual rectangles that we have when dealing with xy-

coordinates. Nonetheless, if our changes in r and θ are small, then our regions will approximate rectangles.

If we look at the shaded region above, we see that one side of it corresponds to the change in radius, Δr, and the other side corresponds to the length of a circular arc where the radius is r and the angle is represented by $\Delta\theta$. In this case, the arc length is $r\Delta\theta$. And since our element of area is approximately rectangular when the changes in our variables are small, we can unequivocally state that,

$$\Delta A \approx \Delta r \cdot r\Delta\theta = r\Delta r\Delta\theta$$

This tells us that the corresponding formula for differentials is,

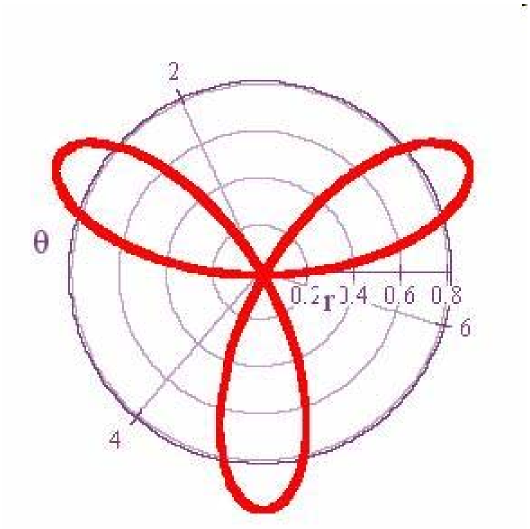
$$dA = r\,drd\theta$$

Hence, if we make this substitution for dA in our double integral, then we get

$$\iint_R f(x,y)\,dA = \int_e^f \int_g^h f(r\cos\theta, r\sin\theta)\,rdrd\theta$$

In other words, to write our double integral as an iterated integral in polar coordinates, we have to replace dA by $r\,drd\theta$, the variable x by $r\cos\theta$, and the variable y by $r\sin\theta$. Additionally, our limits of integration have to be changed to reflect the corresponding intervals for r and θ. Let's look at a few examples.

Example 6: Find the area of one petal of the three petal rose that is the graph of $r = \sin(3\theta)$.



Solution: Let $\begin{cases} z = 1 \\ 0 \le \theta \le \dfrac{\pi}{3} \\ 0 \le r \le \sin(3\theta) \end{cases}$. Then,

$$\text{Area } = \iint_R dA = \int_0^{\pi/3}\int_0^{\sin(3\theta)} r\,dr\,d\theta = \int_0^{\pi/3} \frac{r^2}{2}\bigg|_0^{\sin(3\theta)} d\theta = \int_0^{\pi/3} \frac{\sin^2(3\theta)}{2}\,d\theta$$

$$= \frac{1}{6}\int_0^{\pi} \sin^2 u\,du = \frac{1}{6}\int_0^{\pi} \frac{1-\cos 2u}{2}\,du = \frac{1}{6}\left(\frac{u}{2} - \frac{\sin 2u}{4}\right)\bigg|_0^{\pi} = \frac{\pi}{12}. \quad \blacksquare$$

As you can see, one of the tricks is to simply describe your region properly in terms of polar coordinates. Once you've accomplished that, the other substitutions are usually pretty easy. Now let's look at another example where our polar limits of integration are already given to us.

Example 7: Find the integral of $z = \frac{1}{(x^2+y^2)^{3/2}}$ on the region R corresponding to

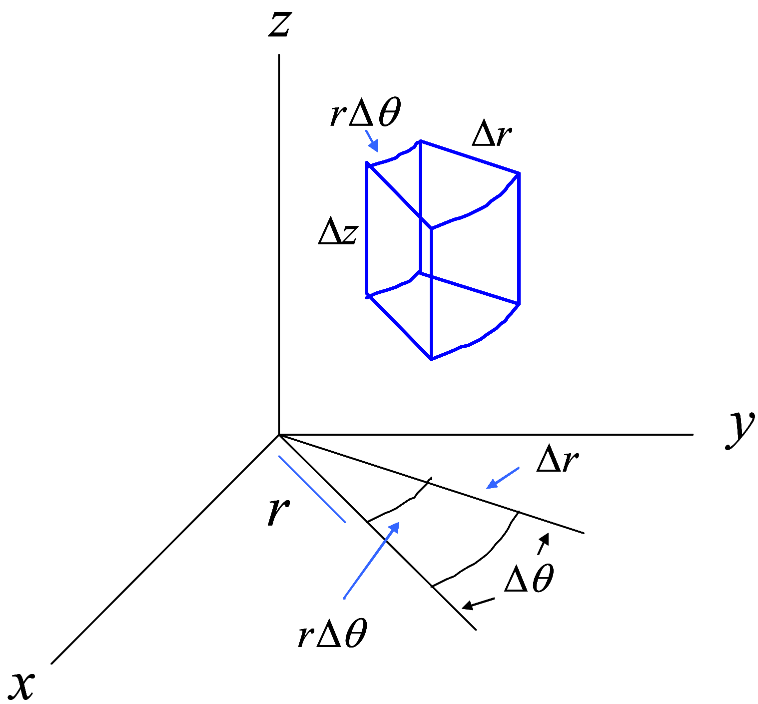
$0 \le \theta \le \frac{\pi}{4}$ and $1 \le r \le 2$.

Solution: Since our integrand involves the expression x^2+y^2, we can immediately replace this by r^2. Thus,

$$\iint_R \frac{1}{(x^2+y^2)^{3/2}}\,dA = \int_0^{\pi/4}\int_1^2 \frac{1}{(r^2)^{3/2}}\,r\,drd\theta = \int_0^{\pi/4}\int_1^2 r^{-2}\,drd\theta$$

$$= \int_0^{\pi/4} -\frac{1}{r}\bigg|_1^2\,d\theta = \int_0^{\pi/4}\frac{1}{2}d\theta = \frac{\theta}{2}\bigg|_0^{\pi/4} = \frac{\pi}{8}. \quad \blacksquare$$

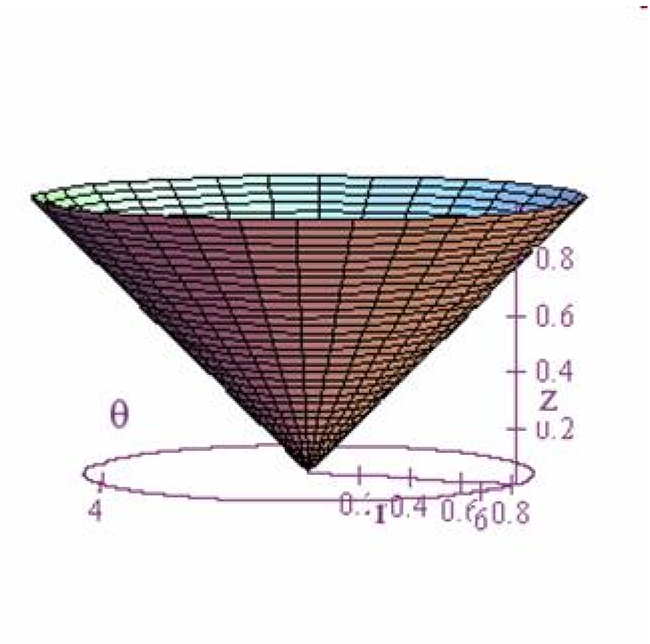
If we are using a triple integral to find a volume and if the corresponding region in the *xy*-plane is easily described by polar coordinates, then it might be a good idea to express the triple integral in cylindrical coordinates. Recall that in cylindrical coordinates, the first two coordinates are polar and the third coordinate is still *z* as in the usual *xyz*-rectangular system. Now to properly convert to cylindrical coordinates, the trick is going to be to figure out how an element of volume would be represented in this coordinate system. The following diagram will guide us in how to do this.



$$\Delta V \approx r\Delta\theta \cdot \Delta z \cdot \Delta r = r\Delta z\Delta r\Delta\theta$$
$$dV = r\,dzdrd\theta$$

Again, when the changes in our variables are small, the element of volume can be approximated by a cube. From the diagram above we see that we can take the height of this cube to be Δz, and the sides of the base to be Δr and $r\Delta\theta$. That results in the approximation we see above that $\Delta V \approx r\Delta\theta\Delta z\Delta r = r\Delta z\Delta r\Delta\theta$. Or as we write in the differential version, $dV = r\,dzdrd\theta$. Now let's take an example or two.

Example 8: Find the volume of the region above the *xy*-plane and beneath the cone defined by $z = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$, $0 \le r \le 1$, and $0 \le \theta \le 2\pi$.



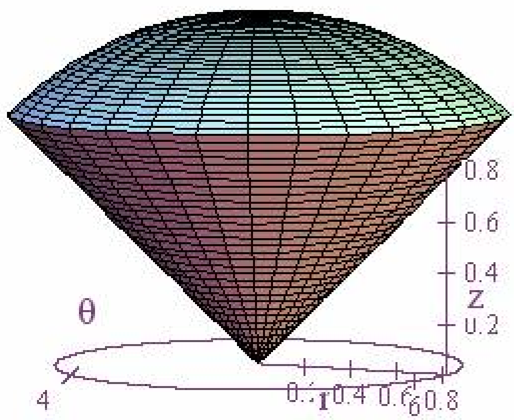
Solution: In this case, the range for *z* will be $0 \le z \le r$. Thus,

$$\text{Volume } = \iiint_V dV = \int_0^{2\pi}\int_0^1\int_0^r r\,dz dr d\theta = \int_0^{2\pi}\int_0^1 rz\Big|_0^r \,dr d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^2\,dr d\theta = \int_0^{2\pi} \frac{r^3}{3}\Bigg|_0^1 d\theta = \int_0^{2\pi} \frac{1}{3} d\theta = \frac{\theta}{3}\Bigg|_0^{2\pi} = \frac{2\pi}{3}. \quad \blacksquare$$

One of my favorite uses of polar coordinates is to find the volume of a filled ice cream cone such as we do in the next example.

Example 9: Find the volume of the ice cream cone defined by $r \le z \le \sqrt{2-r^2}$, $0 \le r \le 1$, and $0 \le \theta \le 2\pi$.



Solution: There are a couple of things we should probably point out before getting started. First, the ranges for the polar coordinates, $0 \le r \le 1$ and $0 \le \theta \le 2\pi$, describe the unit circle in the *xy*-plane, the circle of radius 1 with center at the origin. Second, $z = r$ is going to give us the same cone that we looked at in the previous example. The ice cream that is put in the cone, however, is what we claim is described by $z = \sqrt{2-r^2}$. To see why this is so, recall that $r^2 = x^2 + y^2$. Hence, $z = \sqrt{2-r^2} = \sqrt{2-(x^2+y^2)} \Rightarrow z^2 = 2-(x^2+y^2) \Rightarrow x^2+y^2+z^2 = 2$. This last equation is nothing more than the equation for a sphere of radius $\sqrt{2}$ with center at the origin. However, the restrictions in the *xy*-plane that $0 \le r \le 1$ and $0 \le \theta \le 2\pi$ result in us getting only that portion of the sphere that tops off the ice cream cone. Now let's find the volume.

$$\text{Volume } = \iiint_V dV = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dzdrd\theta = \int_0^{2\pi}\int_0^1 rz\Big|_r^{\sqrt{2-r^2}}\,drd\theta$$

$$= \int_0^{2\pi}\int_0^1\left(r\sqrt{2-r^2}-r^2\right)drd\theta = \int_0^{2\pi}\left(\frac{-(2-r^2)^{3/2}}{3}-\frac{r^3}{3}\right)\Bigg|_0^1 d\theta$$

$$= \int_0^{2\pi}\frac{(2^{3/2}-2)}{3}d\theta = \frac{\theta(2^{3/2}-2)}{3}\Bigg|_0^{2\pi} = \frac{2\pi(2^{3/2}-2)}{3} = \frac{4\pi}{3}\left(\sqrt{2}-1\right). \quad \blacksquare$$

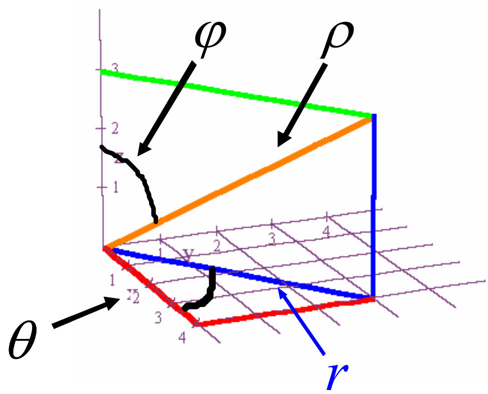
I love combining math with food! Also, note that if we were trying to find this same volume using rectangular coordinates, then our integral would look like this.

$$\text{Volume } = \iiint_V dV = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dzdydx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\left(\sqrt{2-x^2-y^2}-\sqrt{x^2+y^2}\right)dydx$$

Trust me. This integral is not a piece of cake in rectangular coordinates. Changing to cylindrical coordinates makes the problem much easier.

We'll soon redo the same problem in spherical coordinates. First, though, recall some of the basic formulas for converting between rectangular and spherical coordinates.



$$(\rho,\theta,\varphi)$$

$$\rho^2 = x^2 + y^2 + z^2$$

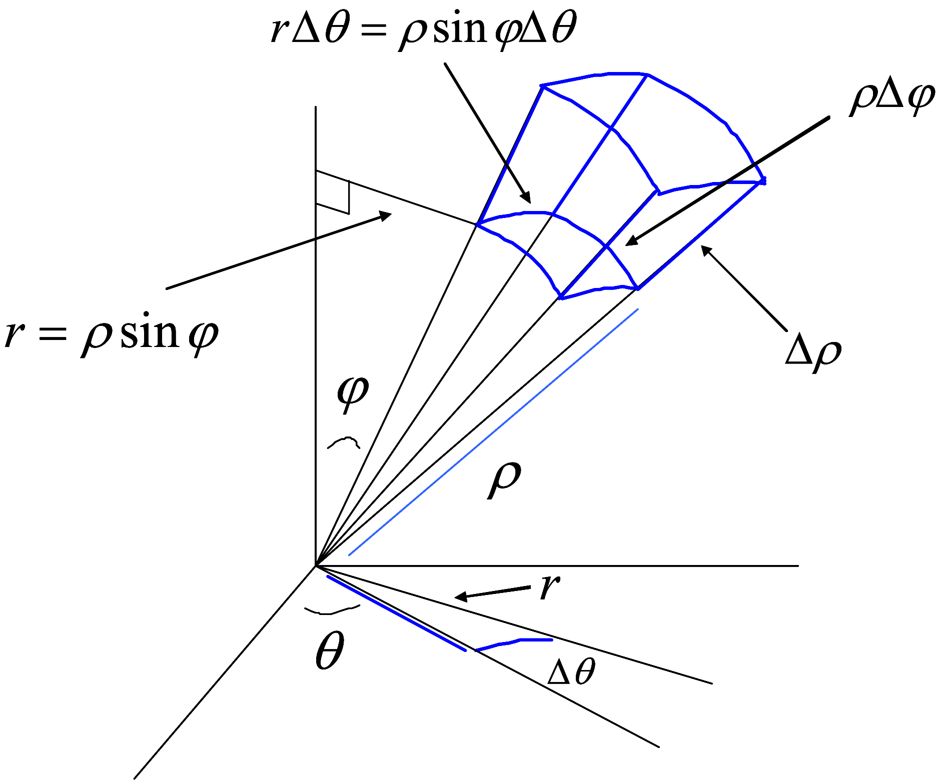
$$\tan\theta = \frac{y}{x}$$

$$\varphi = \arccos\frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$x = r\cos(\theta) = \rho\sin(\varphi)\cos(\theta)$$
$$y = r\sin(\theta) = \rho\sin(\varphi)\sin(\theta)$$
$$z = \rho\cos(\varphi)$$

$$0 \le \rho < \infty$$
$$0 \le \varphi \le \pi$$
$$0 \le \theta < 2\pi$$

Since we've gone over this diagram before, we'll leave it to you to do the math this time. Instead, we'll focus now on how to describe an element of volume in spherical coordinates. As you might expect, this element will be a piece of a sphere, and as the changes in our variables get smaller, this piece will approximate a cube.



$$\Delta V \approx \Delta\rho \cdot \rho\Delta\varphi \cdot \rho\sin\varphi\Delta\theta = \rho^2 \sin\varphi\,\Delta\rho\Delta\varphi\Delta\theta$$

From this diagram we can see that our element of volume is,

$$\Delta V \approx \Delta\rho \cdot \rho\Delta\varphi \cdot r\Delta\theta = \Delta\rho \cdot \rho\Delta\varphi \cdot \rho\sin\varphi\Delta\theta = \rho^2 \sin\varphi\,\Delta\rho\Delta\varphi\Delta\theta$$

In differential form this becomes,

$$dV = \rho^2 \sin\varphi \;\, d\rho \; d\varphi \; d\theta$$

Thus, in spherical coordinates our triple integral for volume becomes the following iterated integral,

$$\text{Volume } = \iiint_V dV = \int_a^b \int_c^d \int_e^f \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

Using spherical coordinates, we can now easily show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Example 10: Find the volume of a sphere of radius r with center at the origin.

Solution: We simply set up a triple integral with the following limits and integrate.

$$0 \le \rho \le r$$
$$0 \le \varphi \le \pi$$
$$0 \le \theta \le 2\pi$$

$$\text{Volume } = \iiint_V dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi} \left.\frac{\rho^3 \sin\varphi}{3}\right|_0^r d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{r^3 \sin\varphi}{3} d\varphi d\theta = \int_0^{2\pi} \left. -\frac{r^3 \cos\varphi}{3}\right|_0^{\pi} d\theta = \int_0^{2\pi} \frac{2r^3}{3} d\theta = \left.\frac{2r^3\theta}{3}\right|_0^{2\pi} = \frac{4}{3}\pi r^3$$

Now we'll use spherical coordinates to tackle the ice cream problem!

Example 11: Find the volume of the ice cream cone defined by $0 \le \theta \le 2\pi$, $0 \le \varphi \le \frac{\pi}{4}$, and $0 \le \rho \le \sqrt{2}$.

Solution: First, notice how incredibly easy it is to describe this solid region in spherical coordinates. Our angle θ in the *xy*-plane goes full circle from 0 to 2π, our angle φ with the positive *z*-axis goes from 0 to $\frac{\pi}{4}$, and the sphere radius goes from 0 to $\sqrt{2}$. It's as easy as π! And now that we understand our limits of integration, we can find the volume.

$$\text{Volume } = \iiint_V dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\sin\varphi \Big|_0^{\sqrt{2}} d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{2^{3/2}}{3}\sin\varphi \, d\varphi d\theta$$

$$= \int_0^{2\pi} \frac{2^{3/2}}{3}(-\cos\varphi)\Big|_0^{\pi/4} d\theta = \int_0^{2\pi} \frac{2^{3/2}}{3}\left(\frac{-1}{\sqrt{2}}+1\right) d\theta$$

$$= \frac{\theta \cdot 2^{3/2}}{3}\left(\frac{-1}{\sqrt{2}}+1\right)\Big|_0^{2\pi} = \frac{2\pi}{3}\cdot 2^{3/2}\left(\frac{-1}{\sqrt{2}}+1\right) = \frac{4\pi}{3}\cdot\sqrt{2}\left(\frac{-1}{\sqrt{2}}+1\right)$$

$$= \frac{4\pi}{3}\left(\sqrt{2}-1\right). \quad \blacksquare$$

Here's another example that's tailor made for spherical coordinates.

Example 12: Find $\iiint_V z^2 \, dV$ on the region between the spheres with radii $\rho = 1$ and $\rho = 2$.

Solution: Notice in this problem that we have $\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \\ 1 \le \rho \le 2 \\ z = \rho\cos\varphi \Rightarrow z^2 = \rho^2\cos^2\varphi \end{cases}$.

Hence,

$$\iiint_V z^2\,dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{2} \rho^2\cos^2\varphi\,\rho^2\sin\varphi\,d\rho d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\left(\frac{\rho^5}{5}\cos^2\varphi\sin\varphi\right)\Bigg|_1^2 d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{31}{5}\cos^2\varphi\sin\varphi\,d\varphi d\theta$$

$$= \int_0^{2\pi}\frac{31}{5}\frac{\left(-\cos^3\varphi\right)}{3}\Bigg|_0^{\pi} d\theta = \int_0^{2\pi}\frac{62}{15}d\theta = \frac{62\theta}{15}\Bigg|_0^{2\pi}$$

$$= \frac{62\cdot 2\pi}{15} - \frac{62\cdot 0}{15} = \frac{124\pi}{15}. \quad \blacksquare$$

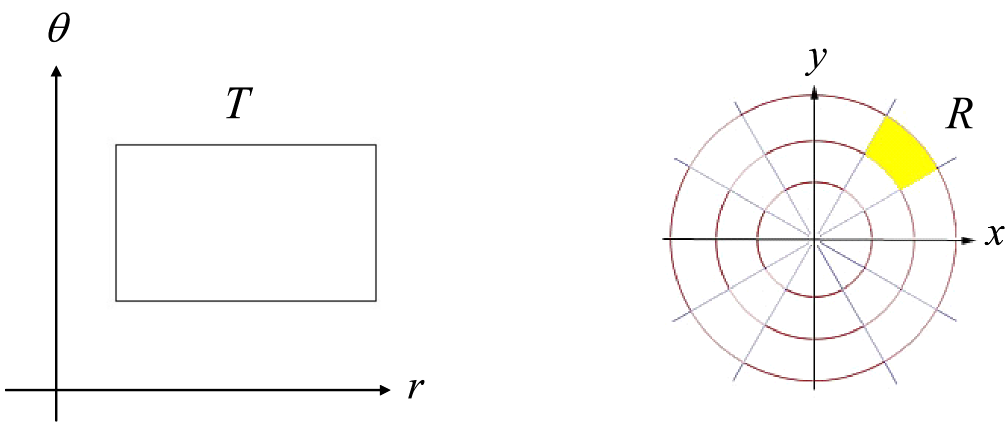
What could be simpler!

The above examples certainly illustrate how changing to a different coordinate system can often make an integration easier to do. However, our limitation has been that we only had a few different coordinate systems to convert to, polar, cylindrical, or spherical. Nonetheless, these examples raise the question of whether it is possible to find a general formula for changing from rectangular coordinates to any coordinate system whatsoever. This last part of our chapter is devoted to accomplishing this task, and to help us develop a more general procedure, we'll take another look at what goes on when we change from rectangular to polar coordinates.

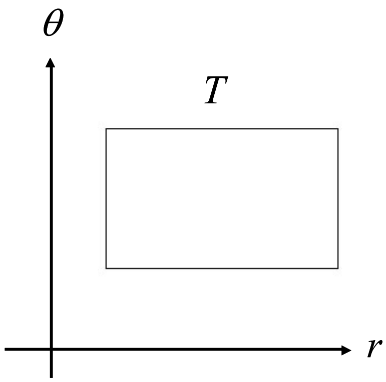
When we convert a double integral from rectangular to polar coordinates, recall the changes that must be made to x, y, and dA.

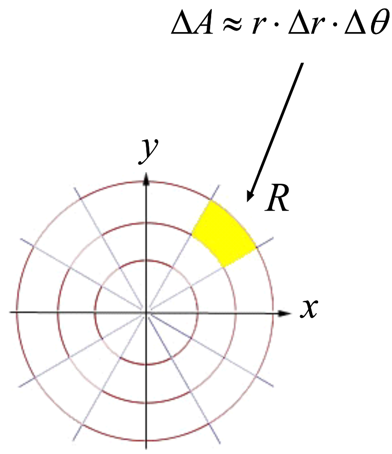
$$x = x(r,\theta) = r\cos\theta$$
$$y = y(r,\theta) = r\sin\theta$$
$$dA = r\,dr d\theta$$

In the polar coordinate system, an element of area is generally a rectangle corresponding to a range of values for r and θ.



However, as you can see above, in the xy-coordinate system, this rectangle takes on a different shape, and the formula for an element of area changes.





And of course, this leads to the following formula for double integrals in polar coordinates.

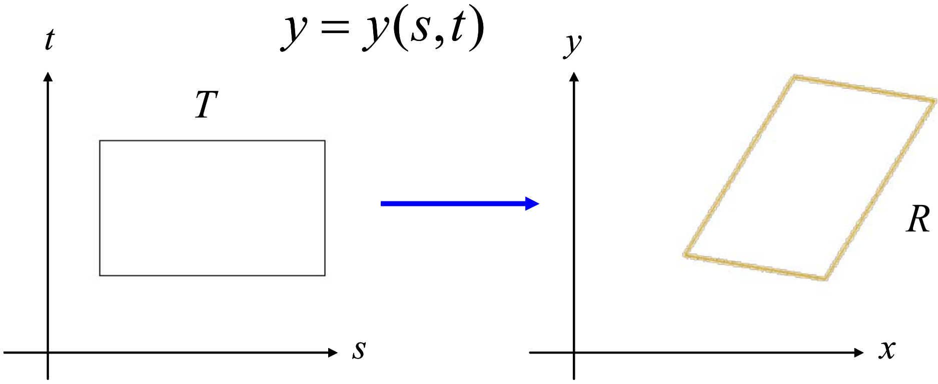
$$\int\int_R f(x,y)\,dA = \int\int_T f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Using the polar coordinate example as a model, we now want to develop a general method for finding change of variable formulas such as the polar coordinate one. Thus, suppose we have a rectangle in an *st*-coordinate system and a pair of functions that converts (s,t) coordinates into (x,y) coordinates.

$$x = x(s,t)$$
$$y = y(s,t)$$

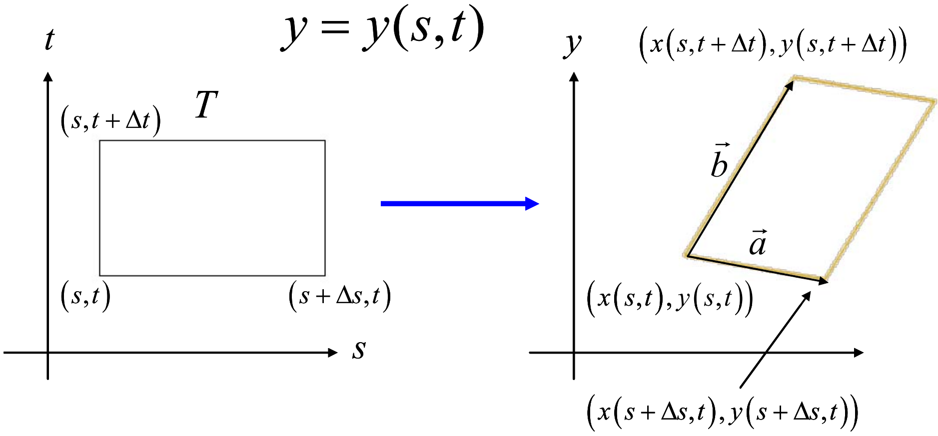
Suppose also that these functions are differentiable and that the transformation from the *st*-coordinates to *xy*-coordinates is one-to-one. Then because of differentiability, local linearity will be present and a small rectangle in the *st*-coordinate system will be mapped onto approximately a parallelogram in the *xy*-coordinate system.

$$x = x(s,t)$$
$$y = y(s,t)$$



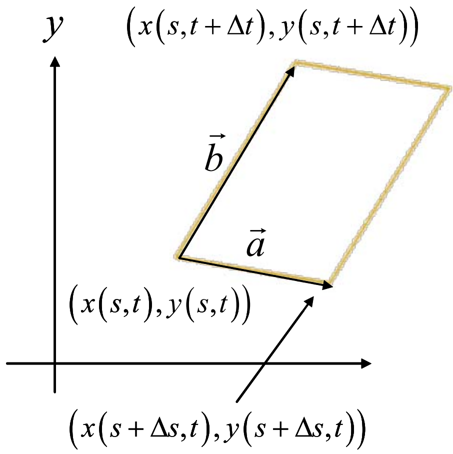
If we add some coordinates, then it looks like this.

$$x = x(s,t)$$
$$y = y(s,t)$$



Since an element of area in our *xy*-coordinate system is represented by a parallelogram, the area of this parallelogram is equal to $\left\|\vec{a}\times\vec{b}\right\|$ where $\vec{a}$ and $\vec{b}$ are vectors that correspond to the two sides of the parallelogram.

$$Area = \left\|\vec{a}\times\vec{b}\right\|$$



However, notice that,

$$\begin{aligned}\vec{a} &= \left(x(s+\Delta s,t)-x(s,t)\right)\hat{i}+\left(y(s+\Delta s,t)-y(s,t)\right)\hat{j}\\ &\approx \frac{\partial x}{\partial s}\Delta s\,\hat{i}+\frac{\partial y}{\partial s}\Delta s\,\hat{j}\end{aligned}$$

$$\begin{aligned}\vec{b} &= \left(x(s,t+\Delta t)-x(s,t)\right)\hat{i}+\left(y(s,t+\Delta t)-y(s,t)\right)\hat{j}\\ &\approx \frac{\partial x}{\partial t}\Delta t\,\hat{i}+\frac{\partial y}{\partial t}\Delta t\,\hat{j}\end{aligned}$$

Hence,

$$\vec{a}\times\vec{b}\approx\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial x}{\partial s}\Delta s & \frac{\partial y}{\partial s}\Delta s & 0\\ \frac{\partial x}{\partial t}\Delta t & \frac{\partial y}{\partial t}\Delta t & 0\end{vmatrix}=\left(\frac{\partial x}{\partial s}\Delta s\cdot\frac{\partial y}{\partial t}\Delta t-\frac{\partial y}{\partial s}\Delta s\cdot\frac{\partial x}{\partial t}\Delta t\right)\hat{k}$$

Furthermore,

$$\left\|\vec{a}\times\vec{b}\right\|\approx\left|\frac{\partial x}{\partial s}\Delta s\cdot\frac{\partial y}{\partial t}\Delta t-\frac{\partial y}{\partial s}\Delta s\cdot\frac{\partial x}{\partial t}\Delta t\right|=\left|\frac{\partial x}{\partial s}\frac{\partial y}{\partial t}-\frac{\partial y}{\partial s}\frac{\partial x}{\partial t}\right|\Delta s\Delta t$$

The expression inside the last absolute value sign is called the *Jacobian*, and it is usually written as,

$$\frac{\partial(x,y)}{\partial(s,t)}=\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}\end{vmatrix}=\frac{\partial x}{\partial s}\frac{\partial y}{\partial t}-\frac{\partial y}{\partial s}\frac{\partial x}{\partial t}$$

Consequently, the area of our parallelogram is equal to the absolute value of the *Jacobian* times the change in *s* and the change in *t*.

$$Area=\Delta A=\left\|\vec{a}\times\vec{b}\right\|\approx\left|\frac{\partial x}{\partial s}\frac{\partial y}{\partial t}-\frac{\partial y}{\partial s}\frac{\partial x}{\partial t}\right|\Delta s\Delta t=\left|\frac{\partial\left(x,y\right)}{\partial\left(s,t\right)}\right|\Delta s\Delta t$$

And this tells us exactly what to substitute for dA in our integral formula,

$$dA = \left| \frac{\partial(x,y)}{\partial(s,t)} \right| dsdt$$

$$\iint_R f(x,y)\,dA = \lim_{\Delta A \to 0} \sum f(x,y) \cdot \Delta A$$
$$= \lim_{\Delta s, \Delta t \to 0} \sum f(x(s,t), y(s,t)) \cdot \left| \frac{\partial(x,y)}{\partial(s,t)} \right| \Delta s \Delta t$$
$$= \iint_T f(x(s,t), y(s,t)) \cdot \left| \frac{\partial(x,y)}{\partial(s,t)} \right| dsdt$$

Now let's verify that this formula works for transformations to polar coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

$$\left| \frac{\partial(x,y)}{\partial(r,\theta)} = \right| = |r| = r$$

$$dA = \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| drd\theta = r\,drd\theta$$

Bingo!

$$\iint_R f(x,y)\,dA = \iint_T f(r\cos\theta, r\sin\theta) \cdot \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| drd\theta$$
$$= \iint_T f(r\cos\theta, r\sin\theta) \cdot r\,drd\theta$$

Now let's try to figure out how to transform an ellipse into a circle. Suppose we start with the equation for an ellipse below, and we'll assume that both *a* and *b* are positive.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

If we set $\begin{cases} x = a\cdot s \\ y = b\cdot t \end{cases}$, then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{(a\cdot s)^2}{a^2}+\frac{(b\cdot t)^2}{b^2}=1 \Rightarrow s^2+t^2=1$$

Hence, the *Jacobian* of this transformation is,

$$\frac{\partial(x,y)}{\partial(s,t)}=\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}=\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}=ab$$

And the absolute value of the *Jacobian* is,

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right|=ab$$

Using this transformation, we can now easily find the area of the ellipse.

$$\begin{aligned} Area\,of\ ellipse &= \iint_{ellipse} dA = \iint_{unit\,circle} ab\,dsdt \\ &= ab\left(\iint_{unit\,circle} dsdt\right) = ab\cdot\pi = \pi ab \end{aligned}$$

It's a piece of lettuce! Nothing to it!

Everything we've done involving changing coordinate systems carries over to higher dimensions, too. In fact, if we have transformation involving three variables, then our *Jacobian* looks like this,

$$\frac{\partial(x,y,z)}{\partial(s,t,u)}=\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u} \end{vmatrix}$$

Here's an example of how find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by first using a change of variables to transform the ellipsoid into a sphere. Again, we will assume that *a*, *b*, and *c* are positive.

Let $\begin{cases} x = a\cdot s \\ y = b\cdot t \\ z = c\cdot u \end{cases}$. Then this will transform our ellipsoid into a sphere of radius 1.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\Rightarrow\frac{a^2s^2}{a^2}+\frac{b^2t^2}{b^2}+\frac{c^2u^2}{c^2}=1\Rightarrow s^2+t^2+u^2=1$$

Furthermore, recall that the volume of a sphere of radius r is given by the formula

$V=\frac{4}{3}\pi r^3$. Hence, in this case with $r=1$, the volume is just $\frac{4\pi}{3}$.

Additionally, the *Jacobian* of this transformation is,

$$\frac{\partial(x,y,z)}{\partial(s,t,u)}=\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u}\\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} & \frac{\partial x}{\partial u}\\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u}\end{vmatrix}=\begin{vmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}=abc$$

Thus, the volume of our ellipsoid is,

$$Volume=\iiint_R dV=\iiint_T\left|\frac{\partial(x,y,z)}{\partial(s,t,u)}\right|dsdtdu=\iiint_T abc\,dsdtdu$$
$$=abc\iiint_T dsdtdu=abc\cdot\frac{4\pi}{3}=\frac{4}{3}\pi abc$$

Any questions?