ANGLES BETWEEN VECTORS - ANSWERS

(1-7) Let $\vec{u} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{v} = \hat{i} - 5\hat{j} + \hat{k}$, and $\vec{w} = -3\hat{i} - 2\hat{j} - 8\hat{k}$. Find the angles between the following vectors. Give your answers in degrees rounded, if necessary, to the nearest tenth of a degree.

1. $\vec{u}$ and $\vec{v}$

$$\theta = \cos^{-1}\left(\frac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) = 108.8°$$

2. $\vec{u}$ and $\vec{w}$

$$\theta = \cos^{-1}\left(\frac{\vec{u}\bullet\vec{w}}{\|\vec{u}\|\|\vec{w}\|}\right) = 158.6°$$

3. $\vec{v}$ and $\vec{w}$

$$\theta = \cos^{-1}\left(\frac{\vec{v}\bullet\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = 91.3°$$

4. $\vec{v}$ and $2\vec{w}$

$$\theta = \cos^{-1}\left(\frac{\vec{v}\bullet 2\vec{w}}{\|\vec{v}\|\|2\vec{w}\|}\right) = 91.3°$$

5. $\vec{v}$ and $\vec{v}$

$$\theta = \cos^{-1}\left(\frac{\vec{v}\bullet\vec{v}}{\|\vec{v}\|\|\vec{v}\|}\right) = 0°$$

6. $\vec{w}$ and $-\vec{w}$

$$\theta = \cos^{-1}\left(\frac{\vec{w}\bullet(-\vec{w})}{\|\vec{w}\|\|-\vec{w}\|}\right) = 180°$$

7. $(\vec{u}+\vec{w})$ and $(\vec{u}-\vec{w})$

$$\theta = \cos^{-1}\left(\frac{(\vec{u}+\vec{w})\bullet(\vec{u}-\vec{w})}{\|(\vec{u}+\vec{w})\|\|(\vec{u}-\vec{w})\|}\right) = 144.3°$$

8. Let $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$ be a nonzero vector, and let α, β, and γ be the angles between $\vec{v}$ and the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$, respectively. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. (NOTE: The angles α, β, and γ are called the *direction angles* of $\vec{v}$, and $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines*.)

Using the dot product, we have that:

$$\cos\alpha = \frac{\hat{i}\bullet\vec{v}}{\|\hat{i}\|\|\vec{v}\|} = \frac{a}{\|\vec{v}\|} = \frac{a}{\sqrt{a^2+b^2+c^2}}$$

$$\cos\beta = \frac{\hat{j}\bullet\vec{v}}{\|\hat{j}\|\|\vec{v}\|} = \frac{b}{\|\vec{v}\|} = \frac{b}{\sqrt{a^2+b^2+c^2}}.$$

$$\cos\gamma = \frac{\hat{k}\bullet\vec{v}}{\|\hat{k}\|\|\vec{v}\|} = \frac{c}{\|\vec{v}\|} = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Hence,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a^2}{a^2+b^2+c^2} + \frac{b^2}{a^2+b^2+c^2} + \frac{c^2}{a^2+b^2+c^2} = \frac{a^2+b^2+c^2}{a^2+b^2+c^2} = 1.$$