ARC LENGTH - ANSWERS

Find the length of the following curves.

1. $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\,\hat{j} + t\hat{k}$, $0 \le t \le 2\pi$

   $\vec{r}\,'(t) = -\sin(t)\hat{i} + \cos(t)\,\hat{j} + \hat{k}$

   $\left\|\vec{r}\,'(t)\right\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

   $L = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\, t\Big|_0^{2\pi} = 2\sqrt{2}\pi$

2. $\vec{r}(t) = t\hat{i} + \dfrac{\sqrt{6}}{2}t^2\,\hat{j} + t^3\hat{k}$, $-1 \le t \le 1$

   $\vec{r}\,'(t) = \hat{i} + \sqrt{6}\,t\,\hat{j} + 3t^2\,\hat{k}$

   $\left\|\vec{r}\,'(t)\right\| = \sqrt{1 + 6t^2 + 9t^4} = 3t^2 + 1$

   $L = \int_{-1}^{1} \left(3t^2 + 1\right) dt = t^3 + t\Big|_{-1}^{1} = 4$

3. $\vec{r}(t) = \cos^3(t)\hat{i} + \sin^3(t)\,\hat{j}$, $0 \le t \le \pi/2$

   $\vec{r}\,'(t) = 3\cos^2 t(-\sin t)\hat{i} + 3\sin^2 t(\cos t)\,\hat{j}$

   $\left\|\vec{r}\,'(t)\right\| = \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t} = 3\cos t\sin t,\ 0 \le t \le \pi/2$

   $L = \int_0^{\pi/2} 3\cos t\sin t\, dt = \dfrac{3\sin^2 t}{2}\Big|_0^{\pi/2} = \dfrac{3}{2}$

4. $\vec{r}(t) = 2(t^2 - 1)^{3/2}\,\hat{i} + 3t^2\,\hat{j} + 3t^2\hat{k}$, $1 \le t \le \sqrt{8}$

   $\vec{r}\,'(t) = 6t\sqrt{t^2 - 1}\,\hat{i} + 6t\,\hat{j} + 6t\,\hat{k}$

   $\left\|\vec{r}\,'(t)\right\| = 6t\sqrt{t^2 + 1},\ 1 \le t \le \sqrt{8}$

   $L = \int_1^{\sqrt{8}} 6t\sqrt{t^2 + 1}\, dt = 54 - 4\sqrt{2}$

5. $\vec{r}(t) = r \cdot \cos(t)\hat{i} + r \cdot \sin(t)\,\hat{j}$, $0 \le t \le 2\pi$ & $r > 0$

   $\vec{r}\,'(t) = -r\sin t\,\hat{i} + r\cos t\,\hat{j}$

   $\left\|\vec{r}\,'(t)\right\| = \sqrt{r^2\sin^2 t + r^2\cos^2 t} = r,\ r > 0$

   $L = \int_0^{2\pi} r\, dt = rt\Big|_0^{2\pi} = 2\pi r$

6. $\vec{r}(t) = 3\cos(2t)\hat{i} + 3\sin(2t)\hat{j} + 3t\hat{k}$, $0 \le t \le \pi/2$

$\vec{r}\,'(t) = -6\sin(2t)\hat{i} + 6\cos(2t)\hat{j} + 3\hat{k}$

$\|\vec{r}\,'(t)\| = \sqrt{36\sin^2 2t + 36\cos^2 2t + 9} = \sqrt{45} = 3\sqrt{5}$

$L = \int_0^{\pi/2} 3\sqrt{5}\, dt = 3\sqrt{5}\, t\Big|_0^{\pi/2} = \dfrac{3\pi\sqrt{5}}{2}$

7. $\vec{r}(t) = (t^2+1)\cos t\,\hat{i} + (t^2+1)\sin t\,\hat{j} + 2t\sqrt{2}\,\hat{k}$, $0 \le t \le 1$

$\vec{r}\,'(t) = \left[2t\cos t - (t^2+1)\sin t\right]\hat{i} + \left[2t\sin t + (t^2+1)\cos t\right]\hat{j} + 2\sqrt{2}\hat{k}$

$\|\vec{r}\,'(t)\| = \sqrt{4t^2\cos^2 t - 4t\cos t(t^2+1)\sin t + (t^4+2t^2+1)\sin^2 t + 4t^2\sin^2 t + 4t\sin t(t^2+1)\cos t + (t^4+2t^2+1)\cos^2 t + 8}$

$= \sqrt{4t^2 + t^4 + 2t^2 + 1 + 8} = \sqrt{t^4 + 6t^2 + 9} = \sqrt{(t^2+3)^2} = t^2 + 3$

$L = \int_0^1 (t^2+3)\, dt = \dfrac{t^3}{3} + 3t\Big|_0^1 = \dfrac{1}{3} + 3 = \dfrac{10}{3}$