CHAIN RULE - ANSWERS

If $x = t^3$ and $y = \sin t$, use the chain rule to find $\frac{dz}{dt}$. Show your work!

$\frac{dx}{dt} = 3t^2$, $\frac{dy}{dt} = \cos t$

1. $z = f(x, y) = x^3 y^2$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 3x^2 y^2 \cdot 3t^2 + 2x^3 y \cdot \cos t$$
$$= 3(t^3)^2 \sin^2 t \cdot 3t^2 + 2(t^3)^3 \sin t \cdot \cos t$$
$$= 9t^8 \sin^2 t + 2t^9 \sin t \cos t$$

2. $z = f(x, y) = \sin(x^3 y^2)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \cos(x^3 y^2) \cdot 3x^2 y^2 \cdot 3t^2 + \cos(x^3 y^2) \cdot 2x^3 y \cdot \cos t$$
$$= \cos((t^3)^3 \sin^2 t) \cdot 3(t^3)^2 \sin^2 t \cdot 3t^2 + \cos((t^3)^3 \sin^2 t) \cdot 2(t^3)^3 \sin t \cdot \cos t$$
$$= \cos\left(t^9 \sin^2 t\right)\left[9t^8 \sin^2 t + 2t^9 \sin t \cos t\right]$$

3. $z = f(x, y) = \sqrt{x^3 y^2}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{1}{2\sqrt{x^3 y^2}} \cdot 3x^2 y^2 \cdot 3t^2 + \frac{1}{2\sqrt{x^3 y^2}} \cdot 2x^3 y \cdot \cos t$$
$$= \frac{1}{2\sqrt{(t^3)^3 \sin^2 t}} \cdot 3(t^3)^2 \sin^2 t \cdot 3t^2 + \frac{1}{2\sqrt{(t^3)^3 \sin^2 t}} \cdot 2(t^3)^3 \sin t \cdot \cos t$$
$$= \frac{1}{2\sqrt{t^9 \sin^2 t}}\left[9t^8 \sin^2 t + 2t^9 \sin t \cos t\right]$$

4. $z = f(x,y) = \sec(x^3y^2)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \sec(x^3y^2)\tan(x^3y^2)\cdot 3x^2y^2\cdot 3t^2 + \sec(x^3y^2)\tan(x^3y^2)\cdot 2x^3y\cdot\cos t$$

$$= \sec((t^3)^3\sin^2 t)\tan((t^3)^3\sin^2 t)\cdot 3(t^3)^2\sin^2 t\cdot 3t^2$$

$$+ \sec((t^3)^3\sin^2 t)\tan((t^3)^3\sin^2 t)\cdot 2(t^3)^3\sin t\cdot\cos t$$

$$= \sec\left(t^9\sin^2 t\right)\tan\left(t^9\sin^2 t\right)\left[9t^8\sin^2 t + 2t^9\sin t\cos t\right]$$

5. $z = f(x,y) = \tan(x^3y^2)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \sec^2(x^3y^2)\cdot 3x^2y^2\cdot 3t^2 + \sec^2(x^3y^2)\cdot 2x^3y\cdot\cos t$$

$$= \sec^2((t^3)^3\sin^2 t)\cdot 3(t^3)^2\sin^2 t\cdot 3t^2 + \sec^2((t^3)^3\sin^2 t)\cdot 2(t^3)^3\sin t\cdot\cos t$$

$$= \sec^2\left(t^9\sin^2 t\right)\left[9t^8\sin^2 t + 2t^9\sin t\cos t\right]$$

6. $z = f(x,y) = \sin^{-1}(x^3y^2)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{1}{\sqrt{1-(x^3y^2)^2}}\cdot 3x^2y^2\cdot 3t^2 + \frac{1}{\sqrt{1-(x^3y^2)^2}}\cdot 2x^3y\cdot\cos t$$

$$= \frac{1}{\sqrt{1-((t^3)^3\sin^2 t)^2}}\cdot 3(t^3)^2\sin^2 t\cdot 3t^2 + \frac{1}{\sqrt{1-((t^3)^3\sin^2 t)^2}}\cdot 2(t^3)^3\sin t\cdot\cos t$$

$$= \frac{1}{\sqrt{1-t^{18}\sin^4 t}}\left[9t^8\sin^2 t + 2t^9\sin t\cos t\right]$$

7. Use the chain rule to find $\dfrac{\partial z}{\partial t}$ for $z = x^2y$, $x = \sin(st)$, and $y = t^2 + s^2$.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = 2xy\cdot\cos(st)\cdot s + x^2\cdot 2t = 2\sin(st)(t^2+s^2)\cos(st)s + \sin^2(st)2t$$

$$= 2\sin(st)[st^2\cos(st) + s^3\cos(st) + t\sin(st)].$$

8. Use the chain rule to find $\frac{\partial z}{\partial s}$ for $z = x^2 y^2$, $x = st$, and $y = t^2 - s^2$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2xy^2 \cdot t + 2x^2 y \cdot (-2s) = 2st(t^2 - s^2)^2 t + 2s^2 t^2 (t^2 - s^2)(-2s)$$
$$= 2st(t^2 - s^2)[(t^2 - s^2)t - 2s^2 t] = 2st(t^2 - s^2)^2 (t^3 - s^2 t - 2s^2 t) = 2st(t^2 - s^2)^2 (t^3 - 3s^2 t)$$
$$= 2st^2 (t^2 - s^2)^2 (t^2 - 3s^2).$$

9. If $E = IR$ (voltage = current $\times$ resistance), and if all of these quantities are changing over time *t*, then use the chain rule to write down a formula for the rate at which voltage changes over time.

$$\frac{dE}{dt} = \frac{\partial E}{\partial I}\frac{dI}{dt} + \frac{\partial E}{\partial R}\frac{dR}{dt} = R\frac{dI}{dt} + I\frac{dR}{dt}.$$