CHANGE OF VARIABLES - ANSWERS

1. Find the Jacobian of the following transformation.

$x = 2u - 3v$

$y = u + 2v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = (2)(2) - (1)(-3) = 4 + 3 = 7$$

2. Find the Jacobian of the following transformation.

$x = uv$

$y = 4u^2 + 2v^2$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ 8u & 4v \end{vmatrix} = 4v^2 - 8u^2$$

3. Find the Jacobian of the following transformation.

$x = 2u + v - w$

$y = 3u + 2v + 2w$

$z = u + v + w$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= 2(2\cdot 1 - 1\cdot 2) - 1(3\cdot 1 - 1\cdot 2) - 1(3\cdot 1 - 1\cdot 2) = 0 - 1 - 1 = -2$$

4. Find the area of the ellipse by using a change of variables to transform the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ into a circle.

$$\begin{matrix} x = 2u \\ y = 3v \end{matrix} \Rightarrow \frac{4u^2}{4} + \frac{9v^2}{9} = u^2 + v^2 = 1 \;\&\; \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6 \Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = |6| = 6$$

$$\iint_R dA = \iint_T \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \iint_T 6\, dudv = 6\iint_T dudv = 6\pi$$

5. Find the area of the ellipse by using a change of variables to transform the ellipse $\frac{x^2}{36}+16\frac{y^2}{9}=1$ into a circle

$$\begin{aligned} x &= 6u \\ y &= \frac{3}{4}v \end{aligned} \Rightarrow \frac{36u^2}{36}+16\frac{\frac{9}{16}v^2}{9}=u^2+v^2=1 \;\&\; \frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 6 & 0 \\ 0 & \frac{3}{4} \end{vmatrix}=\frac{9}{2} \Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left|\frac{9}{2}\right|=\frac{9}{2}$$

$$\iint_R dA=\iint_T \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv=\iint_T \frac{9}{2}dudv=\frac{9}{2}\iint_T dudv=\frac{9}{2}\pi$$

6. Find the volume of the ellipsoid by using a change of variables to transform the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}=1$ into a sphere.

$$\begin{aligned} x &= 2u \\ y &= 3v \\ z &= 5w \end{aligned} \Rightarrow \frac{(2u)^2}{4}+\frac{(3v)^2}{9}+\frac{(5w)^2}{25}=u^2+v^2+w^2=1 \;\&\; \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|=\left\|\begin{matrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{matrix}\right\|=|30|=30$$

$$\iiint_V dV=\iiint_T \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw=\iiint_T 30\,dudvdw=30\iiint_T dudvdw=30\cdot\frac{4}{3}\pi=40\pi$$

7. Find the area of the parallelogram with vertices $(0,0),(1,0),(1,1),\&(2,1)$ by using linear equations and a change of variables to transform the parallelogram into a rectangle.

To transform our parallelogram into a rectangle, we will need the following correspondence between vertices in *xy* and vertices in *uv*.

$$(0,0)_{xy} \leftrightarrow (0,0)_{uv}$$
$$(1,0)_{xy} \leftrightarrow (1,0)_{uv}$$
$$(1,1)_{xy} \leftrightarrow (0,1)_{uv}$$
$$(2,1)_{xy} \leftrightarrow (1,1)_{uv}$$

We'll now pull a linear algebra rabbit out of our hat and just assume that what we need to find are equations that express *x* and *y* as linear combinations of *u* and *v*. We know this will work because linear algebra (which you may not have taken yet) tells us that such linear combinations will transform a straight line in the *uv*-plane into straight lines in the *xy*-plane and vice-versa. Thus, if we want $au + bv = x$, then using the last two pairs of coordinates above we get the following equations.

$$\begin{matrix} a \cdot 0 + b \cdot 1 = 1 \\ a \cdot 1 + b \cdot 1 = 2 \end{matrix} \Rightarrow \begin{matrix} a = 1 \\ b = 1 \end{matrix} \Rightarrow x = u + v$$

Similarly, if we want $cu + dv = y$, then

$$\begin{matrix} c \cdot 0 + d \cdot 1 = 1 \\ c \cdot 1 + d \cdot 1 = 1 \end{matrix} \Rightarrow \begin{matrix} c = 0 \\ d = 1 \end{matrix} \Rightarrow y = v$$

Hence,

$$\iint_R dA = \iint_T \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dvdu = \iint_T dvdu = 1$$