CYLINDRICAL INTEGRALS - ANSWERS

For each problem below, set up and evaluate a triple integral in cylindrical coordinates.

1. Use a triple integral in cylindrical coordinates to find the volume of a cylinder of height H and radius R.

$$\int_0^{2\pi}\int_0^R\int_0^H r\,dzdrd\theta = \int_0^{2\pi}\int_0^R rz\Big|_0^H drd\theta = \int_0^{2\pi}\int_0^R rH\,drd\theta = \int_0^{2\pi}\frac{r^2H}{2}\Big|_0^R d\theta$$

$$= \int_0^{2\pi}\frac{R^2H}{2}d\theta = \frac{R^2H}{2}\theta\Big|_0^{2\pi} = \pi R^2 H$$

2. Let V be a sphere with center at the origin and $\text{radius} = R$. Find the volume of V.

$$2\int_0^{2\pi}\int_0^R\int_0^{\sqrt{R^2-r^2}} r\,dzdrd\theta = 2\int_0^{2\pi}\int_0^R rz\Big|_0^{\sqrt{R^2-r^2}} drd\theta = 2\int_0^{2\pi}\int_0^R r\sqrt{R^2-r^2}\,drd\theta$$

$$= -\int_0^{2\pi}\int_{R^2}^0 u^{1/2}\,dud\theta = \int_0^{2\pi}\int_0^{R^2} u^{1/2}\,dud\theta = \int_0^{2\pi}\frac{2u^{3/2}}{3}\Big|_0^{R^2} d\theta = \int_0^{2\pi}\frac{2R^3}{3}d\theta$$

$$= \frac{2R^3}{3}\theta\Big|_0^{2\pi} = \frac{4}{3}\pi R^3$$

3. Find the volume of the solid bounded above by the sphere $x^2+y^2+z^2=1$ and below the cone $z=\sqrt{x^2+y^2}$.

$$\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_r^{\sqrt{1-r^2}} r\,dzdrd\theta = \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} rz\Big|_r^{\sqrt{1-r^2}} drd\theta = \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\left(r\sqrt{1-r^2}-r^2\right)drd\theta$$

$$= \int_0^{2\pi}\left(\frac{-(1-r^2)^{3/2}}{3}-\frac{r^3}{3}\right)\Bigg|_0^{\frac{1}{\sqrt{2}}} d\theta = \int_0^{2\pi}\left(\frac{-\left(1-\frac{1}{2}\right)^{3/2}}{3}-\frac{\frac{1}{2^{3/2}}}{3}+\frac{1}{3}\right)d\theta$$

$$= \int_0^{2\pi}\left(-\frac{1}{3\sqrt{2}}+\frac{1}{3}\right)d\theta = \left(\frac{1}{3}-\frac{1}{3\sqrt{2}}\right)2\pi$$

4. Find the volume of the solid bounded above by $z = -x^2 - y^2 + 2$ and below by the *xy*-plane.

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_0^{-r^2+2} r\,dzdrd\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} rz\Big|_0^{-r^2+2} d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}}\left(-r^3+2r\right)drd\theta$$

$$= \int_0^{2\pi}\left(\frac{-r^4}{4}+r^2\right)\Bigg|_0^{\sqrt{2}} d\theta = \int_0^{2\pi}\left(-1+2\right)d\theta = \int_0^{2\pi} d\theta = 2\pi$$

5. Find the volume of the solid bounded above by $z = -x^2 - y^2 + 1$ and below by $z = x^2 + y^2 - 1$.

$$\int_0^{2\pi}\int_0^{1}\int_{r^2-1}^{-r^2+1} r\,dzdrd\theta = \int_0^{2\pi}\int_0^{1} rz\Big|_{r^2-1}^{-r^2+1} drd\theta = \int_0^{2\pi}\int_0^{1}\left(-r^3+r\right)-\left(r^3-r\right)drd\theta$$

$$= \int_0^{2\pi}\int_0^{1}\left(-2r^3+2r\right)drd\theta = \int_0^{2\pi}\left(\frac{-2r^4}{4}+r^2\right)\Bigg|_0^{1} d\theta = \int_0^{2\pi}\left(-\frac{1}{2}+1\right)d\theta$$

$$= \int_0^{2\pi}\frac{1}{2}d\theta = \frac{\theta}{2}\Big|_0^{2\pi} = \pi$$

6. Find the volume inside the cone $z = \sqrt{x^2 + y^2}$ for $0 \le z \le 3$

$$\int_0^{2\pi}\int_0^{3}\int_r^{3} r\,dzdrd\theta = \int_0^{2\pi}\int_0^{3} rz\Big|_r^{3} drd\theta = \int_0^{2\pi}\int_0^{3}\left(3r-r^2\right)drd\theta$$

$$= \int_0^{2\pi}\left(\frac{3r^2}{2}-\frac{r^3}{3}\right)\Bigg|_0^{3} d\theta = \int_0^{2\pi}\left(\frac{27}{2}-9\right)d\theta$$

$$= \int_0^{2\pi}\frac{9}{2}d\theta = \frac{9\theta}{2}\Big|_0^{2\pi} = 9\pi$$

7. Suppose you drill a hole of radius 1 through the center of a sphere of radius 3. Find the volume of the portion removed by the drill.

$$2\int_0^{2\pi}\int_0^1\int_0^{\sqrt{9-r^2}} r\,dzdrd\theta = 2\int_0^{2\pi}\int_0^1 rz\Big|_0^{\sqrt{9-r^2}}\,drd\theta = 2\int_0^{2\pi}\int_0^1 r\sqrt{9-r^2}\,drd\theta$$

$$= -\int_0^{2\pi} \frac{2(9-r^2)^{3/2}}{3}\Bigg|_0^1 d\theta = \int_0^{2\pi}\left(18-\frac{2^{11/2}}{3}\right)d\theta = \left(18-\frac{2^{11/2}}{3}\right)\theta\Bigg|_0^{2\pi}$$

$$= \left(18-\frac{2^{11/2}}{3}\right)\cdot 2\pi$$