DIRECTIONAL DERIVATIVES - ANSWERS

(1-8) For each of the following functions, find the directional derivative at the point $(1,1)$ in the direction $\vec{u} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$. For problems 1 through 3, also find the maximum and minimum values at $(1,1)$ for any directional derivative. If necessary, round to four decimal places.

$D_{\vec{u}}\, f(1,1) = \nabla f(1,1) \bullet \vec{u}$

1. $z = f(x,y) = x^3y^2$

$$\nabla f = \frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} = 3x^2y^2\,\hat{i} + 2x^3y\,\hat{j} = \left\langle 3x^2y^2, 2x^3y \right\rangle$$

$$\nabla f(1,1) = \langle 3,2 \rangle$$

$$D_{\vec{u}}\, f(1,1) = \nabla f(1,1) \bullet \vec{u} = \langle 3,2 \rangle \bullet \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}} \approx 3.5355$$

$$\max = \left\| \nabla f(1,1) \right\| = \left\| \langle 3,2 \rangle \right\| = \sqrt{13} \approx 3.6056$$

$$\min = -\left\| \nabla f(1,1) \right\| \approx -3.6056$$

2. $z = f(x,y) = \sin(x^3y^2)$

$$\nabla f = \frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} = \cos(x^3y^2)\cdot 3x^2y^2\,\hat{i} + \cos(x^3y^2)\cdot 2x^3y\,\hat{j}$$

$$= \left\langle 3\cos(x^3y^2)x^2y^2, 2\cos(x^3y^2)x^3y \right\rangle$$

$$\nabla f(1,1) = \langle 3\cos(1), 2\cos(1) \rangle$$

$$D_{\vec{u}}\, f(1,1) = \nabla f(1,1) \bullet \vec{u} = \langle 3\cos(1), 2\cos(1) \rangle \bullet \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{3\cos(1)}{\sqrt{2}} + \frac{2\cos(1)}{\sqrt{2}} = \frac{5\cos(1)}{\sqrt{2}} \approx 1.9103$$

$$\max = \left\| \nabla f(1,1) \right\| = \left\| \langle 3\cos(1), 2\cos(1) \rangle \right\| \approx 1.9481$$

$$\min = -\left\| \nabla f(1,1) \right\| \approx -1.9481$$

3. $z = f(x,y) = \sqrt{x^3 y^2}$

$$\nabla f = \frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} = \frac{1}{2\sqrt{x^3y^2}} \cdot 3x^2y^2\,\hat{i} + \frac{1}{2\sqrt{x^3y^2}} \cdot 2x^3y\,\hat{j}$$

$$= \left\langle 3\frac{1}{2\sqrt{x^3y^2}}x^2y^2, 2\frac{1}{2\sqrt{x^3y^2}}x^3y \right\rangle = \left\langle \frac{3x^2y^2}{2\sqrt{x^3y^2}}, \frac{x^3y}{\sqrt{x^3y^2}} \right\rangle$$

$$\nabla f(1,1) = \left\langle \frac{3}{2}, 1 \right\rangle$$

$$D_{\vec{u}} f(1,1) = \nabla f(1,1) \bullet \vec{u} = \left\langle \frac{3}{2}, 1 \right\rangle \bullet \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{3}{2\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{5}{2\sqrt{2}} \approx 1.7678$$

$$\max = \|\nabla f(1,1)\| = \left\| \left\langle \frac{3}{2}, 1 \right\rangle \right\| = \sqrt{\frac{13}{4}} \approx 1.8028$$

$$\min = -\|\nabla f(1,1)\| \approx -1.8028$$

4. $z = f(x,y) = \sec(x^3y^2)$

$$\nabla f = \frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} = \sec(x^3y^2)\tan(x^3y^2) \cdot 3x^2y^2\,\hat{i} + \sec(x^3y^2)\tan(x^3y^2) \cdot 2x^3y\,\hat{j}$$

$$= \left\langle 3\sec(x^3y^2)\tan(x^3y^2)x^2y^2, 2\sec(x^3y^2)\tan(x^3y^2)x^3y \right\rangle$$

$$\nabla f(1,1) = \left\langle 3\sec(1)\tan(1), 2\sec(1)\tan(1) \right\rangle$$

$$D_{\vec{u}} f(1,1) = \nabla f(1,1) \bullet \vec{u} = \left\langle 3\sec(1)\tan(1), 2\sec(1)\tan(1) \right\rangle \bullet \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$= \frac{3\sec(1)\tan(1)}{\sqrt{2}} + \frac{2\sec(1)\tan(1)}{\sqrt{2}} = \frac{5\sec(1)\tan(1)}{\sqrt{2}} \approx 10.1911$$

5. $z = f(x,y) = \tan(x^3y^2)$

$$\nabla f = \frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} = \sec^2(x^3y^2) \cdot 3x^2y^2\,\hat{i} + \sec^2(x^3y^2) \cdot 2x^3y\,\hat{j}$$

$$= \left\langle 3\sec^2(x^3y^2)x^2y^2, 2\sec^2(x^3y^2)x^3y \right\rangle$$

$$\nabla f(1,1) = \left\langle 3\sec^2(1), 2\sec^2(1) \right\rangle$$

$$D_{\vec{u}} f(1,1) = \nabla f(1,1) \bullet \vec{u} = \left\langle 3\sec^2(1), 2\sec^2(1) \right\rangle \bullet \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$= \frac{3\sec^2(1)}{\sqrt{2}} + \frac{2\sec^2(1)}{\sqrt{2}} = \frac{5\sec^2(1)}{\sqrt{2}} \approx 12.1110$$

6. $z = f(x,y) = \sin^{-1}(x^3y^2)$

$$\nabla f = \frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} = \frac{1}{\sqrt{1-(x^3y^2)^2}} \cdot 3x^2y^2\,\hat{i} + \frac{1}{\sqrt{1-(x^3y^2)^2}} \cdot 2x^3y\,\hat{j}$$

$$= \left\langle 3\frac{1}{\sqrt{1-x^6y^4}}x^2y^2, 2\frac{1}{\sqrt{1-x^6y^4}}x^3y \right\rangle = \left\langle \frac{3x^2y^2}{\sqrt{1-x^6y^4}}, \frac{2x^3y}{\sqrt{1-x^6y^4}} \right\rangle$$

$\nabla f(1,1) =$ undefined

$D_{\vec{u}} f(1,1) =$ undefined

7. $z = f(x,y) = \dfrac{1}{x^2+y^2}$

$$\nabla f = \frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} = -(x^2+y^2)^{-2} \cdot 2x\,\hat{i} + -(x^2+y^2)^{-2} \cdot 2y\,\hat{j}$$

$$= \left\langle -\frac{2x}{(x^2+y^2)^2}, -\frac{2y}{(x^2+y^2)^2} \right\rangle$$

$$\nabla f(1,1) = \left\langle -\frac{1}{2}, -\frac{1}{2} \right\rangle$$

$$D_{\vec{u}} f(1,1) = \nabla f(1,1) \bullet \vec{u} = \left\langle -\frac{1}{2}, -\frac{1}{2} \right\rangle \bullet \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$= \frac{-1}{2\sqrt{2}} + \frac{-1}{2\sqrt{2}} = \frac{-2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}} \approx -0.7071$$

8. $z = f(x,y) = x^2e^y$

$$\nabla f = \frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} = 2xe^y\,\hat{i} + x^2e^y\,\hat{j}$$

$$= \left\langle 2xe^y, x^2e^y \right\rangle$$

$$\nabla f(1,1) = \left\langle 2e, e \right\rangle$$

$$D_{\vec{u}} f(1,1) = \nabla f(1,1) \bullet \vec{u} = \left\langle 2e, e \right\rangle \bullet \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$= \frac{2e}{\sqrt{2}} + \frac{e}{e\sqrt{2}} = \frac{3e}{\sqrt{2}} \approx 5.7663$$

9. $w = f(x,y,z) = \sin(xyz)$

$$\nabla f = \frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k} = \cos(xyz)\cdot yz\,\hat{i} + \cos(xyz)\cdot xz\,\hat{j} + \cos(xyz)\cdot xy\,\hat{k}$$

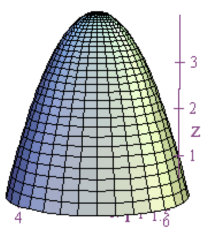
$$= \langle \cos(xyz)\cdot yz, \cos(xyz)\cdot xz, \cos(xyz)\cdot xy \rangle$$

$$\nabla f(1,1,1) = \langle \cos(1), \cos(1), \cos(1) \rangle$$

$$D_{\vec{u}} f(1,1,1) = \nabla f(1,1,1)\bullet\vec{u} = \langle \cos(1), \cos(1), \cos(1) \rangle \bullet \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$= \frac{\cos(1)}{\sqrt{3}} + \frac{\cos(1)}{\sqrt{3}} + \frac{\cos(1)}{\sqrt{3}} = \frac{3\cos(1)}{\sqrt{3}} = \sqrt{3}\cdot\cos(1) \approx 0.9358$$

10. Suppose you are climbing a hill that is in the shape of the graph of $z = -x^2 - y^2 + 4$, and you are standing at the coordinates $\left(\sqrt{2}, \sqrt{2}\right)$. In terms of unit vectors, what direction should your proceed in if you want to ascend the hill most rapidly? What two directions could you walk in to keep your elevation constant?



The obvious answer is to walk in the compass direction that leads straight to the origin. As a unit vector, this direction would be $\vec{u} = -\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$. And to walk around the hill without changing elevation, we could walk in the direction of either of these two vectors, $\vec{v} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$ or $\vec{w} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$.

To show this a little more formally, the gradient of our surface is $\nabla z = -2x\hat{i} - 2y\hat{j}$, and $\nabla z(\sqrt{2},\sqrt{2}) = -2\sqrt{2}\,\hat{i} - 2\sqrt{2}\,\hat{j}$, and $\frac{\nabla z(\sqrt{2},\sqrt{2})}{\left\|\nabla z(\sqrt{2},\sqrt{2})\right\|} = \frac{-2\sqrt{2}\,\hat{i} - 2\sqrt{2}\,\hat{j}}{\sqrt{16}} = \frac{-2\sqrt{2}\,\hat{i} - 2\sqrt{2}\,\hat{j}}{4}$

$= -\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$. Furthermore, one can easily verify that $D_{\vec{v}} z(\sqrt{2},\sqrt{2}) = 0 = D_{\vec{w}} z(\sqrt{2},\sqrt{2})$, showing that there is no change in *z* if we move in the direction of $\vec{v}$ or $\vec{w}$

11. You are walking on a surface in the desert, and being the mathematician you are, you set up an *xyz*-coordinate system with yourself at the origin and you realize that you can model the terrain by $z = 3x - 2y$. Furthermore, so that you can get some exercise without over exerting yourself, you don't want to walk uphill and you don't want to walk downhill. What direction(s) should you proceed in so that your elevation remains constant? Give your answer(s) in terms of an angle with respect to the positive *x*-axis rounded to the nearest tenth of a degree. Additionally, find the unit vector(s) corresponding to your angle(s), and round the components to the nearest hundredth.

    The gradient of $z = 3x - 2y$ is $\nabla z = 3\hat{i} - 2\hat{j}$, and we want to find a direction such that the value of the directional derivative is zero. Also, we can denote our unit vector by $\vec{u} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$. Hence, $0 = \nabla z \bullet \vec{u} = 3\cos\theta - 2\sin\theta \Rightarrow \tan\theta = 3/2 \Rightarrow \theta \approx 56.3°$. However, we can get a second angle by adding $180°$ to the first angle to obtain $236.3°$. The corresponding unit vectors are $\vec{u}_1 = \cos 56.3\hat{i} + \sin 56.3\,\hat{j} = .55\hat{i} + .83\,\hat{j}$ and $\vec{u}_2 = \cos 236.3\hat{i} + \sin 236.3\,\hat{j} = -.55\hat{i} - .83\,\hat{j}$.