GRADIENTS AND TANGENT PLANES - ANSWERS

(1-6) For each of the following functions, use a gradient vector to find the equation of the tangent plane at the point $\left(1,1,f(1,1)\right)$.

1. $z = f(x,y) = x^3y^2$

$$P = \left(1,1,1\right)$$
$$w = x^3y^2 - z$$
$$\nabla w = \frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k} = 3x^2y^2\,\hat{i} + 2x^3y\,\hat{j} - \hat{k} = \left\langle 3x^2y^2, 2x^3y, -1\right\rangle$$
$$\nabla w(1,1,1) = \left\langle 3,2,-1\right\rangle$$
$$3(x-1) + 2(y-1) - 1(z-1) = 0 \Rightarrow 3x + 2y - z - 4 = 0$$
$$\Rightarrow z = 3x + 2y - 4$$

2. $z = f(x,y) = \sin(x^3y^2)$

$$P = \left(1,1,\sin(1)\right) \approx \left(1,1,.84\right)$$
$$w = f(x,y) - z = \sin(x^3y^2) - z$$
$$\nabla w = \frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k} = \cos(x^3y^2)\cdot 3x^2y^2\,\hat{i} + \cos(x^3y^2)\cdot 2x^3y\,\hat{j} - \hat{k}$$
$$= \left\langle 3\cos(x^3y^2)x^2y^2, 2\cos(x^3y^2)x^3y, -1\right\rangle$$
$$\nabla w(1,1,.84) = \left\langle 3\cos(1), 2\cos(1), -1\right\rangle \approx \left\langle 1.62, 1.08, -1\right\rangle$$
$$1.62(x-1) + 1.08(y-1) - 1(z-.84) = 0 \Rightarrow 1.62x + 1.08y - z - 1.86 = 0$$
$$\Rightarrow z = 1.62x + 1.08y - 1.86$$

3. $z = f(x,y) = \sqrt{x^3 y^2}$

$$P = (1,1,1)$$
$$w = f(x,y) - z = \sqrt{x^3 y^2} - z$$
$$\nabla w = \frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k} = \frac{1}{2\sqrt{x^3 y^2}} \cdot 3x^2 y^2\,\hat{i} + \frac{1}{2\sqrt{x^3 y^2}} \cdot 2x^3 y\,\hat{j} - \hat{k}$$
$$= \left\langle 3\frac{1}{2\sqrt{x^3 y^2}} x^2 y^2, 2\frac{1}{2\sqrt{x^3 y^2}} x^3 y, -1 \right\rangle = \left\langle \frac{3x^2 y^2}{2\sqrt{x^3 y^2}}, \frac{x^3 y}{\sqrt{x^3 y^2}}, -1 \right\rangle$$
$$\nabla w(1,1,1) = \left\langle \frac{3}{2}, 1, -1 \right\rangle$$
$$\frac{3}{2}(x-1) + 1(y-1) - 1(z-1) = 0 \Rightarrow \frac{3}{2}x + y - z - \frac{3}{2} = 0$$
$$\Rightarrow z = \frac{3}{2}x + y - \frac{3}{2}$$

4. $z = f(x,y) = \sec(x^3 y^2)$

$$P = (1,1,\sec(1)) \approx (1,1,1.85)$$
$$w = f(x,y) - z = \sec(x^3 y^2) - z$$
$$\nabla w = \frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k} = \sec(x^3 y^2)\tan(x^3 y^2) \cdot 3x^2 y^2\,\hat{i} + \sec(x^3 y^2)\tan(x^3 y^2) \cdot 2x^3 y\,\hat{j} - \hat{k}$$
$$= \left\langle 3\sec(x^3 y^2)\tan(x^3 y^2) x^2 y^2, 2\sec(x^3 y^2)\tan(x^3 y^2) x^3 y, -1 \right\rangle$$
$$\nabla w(1,1,1.85) = \left\langle 3\sec(1)\tan(1), 2\sec(1)\tan(1), -1 \right\rangle \approx \left\langle 8.65, 5.76, -1 \right\rangle$$
$$8.65(x-1) + 5.76(y-1) - 1(z-1.85) = 0 \Rightarrow 8.65x + 5.76y - z - 12.56 = 0$$
$$\Rightarrow z = 8.65x + 5.76y - 12.56$$

5. $z = f(x,y) = \tan(x^3 y^2)$

$$P = (1,1,\tan(1)) \approx (1,1,1.56)$$
$$w = f(x,y) - z = \tan(x^3 y^2) - z$$
$$\nabla w = \frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k} = \sec^2(x^3 y^2) \cdot 3x^2 y^2\,\hat{i} + \sec^2(x^3 y^2) \cdot 2x^3 y\,\hat{j} - \hat{k}$$
$$= \left\langle 3\sec^2(x^3 y^2) x^2 y^2, 2\sec^2(x^3 y^2) x^3 y, -1 \right\rangle$$
$$\nabla w(1,1,1.56) = \left\langle 3\sec^2(1), 2\sec^2(1), -1 \right\rangle \approx \left\langle 10.28, 6.85, -1 \right\rangle$$
$$10.28(x-1) + 6.85(y-1) - 1(z-1.56) = 0 \Rightarrow 10.28x + 6.85y - z - 15.57 = 0$$
$$\Rightarrow z = 10.28x + 6.85y - 15.57$$

6. $z = f(x, y) = \sin^{-1}(x^3 y^2)$

$P = \left(1,1,\sin^{-1}(1)\right) \approx \left(1,1,1.57\right)$

$w = f(x, y) - z = \sin^{-1}(x^3 y^2) - z$

$\nabla w = \frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k} = \frac{1}{\sqrt{1-(x^3y^2)^2}} \cdot 3x^2y^2\,\hat{i} + \frac{1}{\sqrt{1-(x^3y^2)^2}} \cdot 2x^3y\,\hat{j} - \hat{k}$

$= \left\langle 3\frac{1}{\sqrt{1-x^6y^4}}x^2y^2, 2\frac{1}{\sqrt{1-x^6y^4}}x^3y, -1 \right\rangle = \left\langle \frac{3x^2y^2}{\sqrt{1-x^6y^4}}, \frac{2x^3y}{\sqrt{1-x^6y^4}}, -1 \right\rangle$

$\nabla w(1,1,1.57) =$ undefined

7. Find the equation of the tangent plane to the surface $x^2 + y^2 + z^2 = 9$ at the point $P = (0,0,3)$.

Let $w = x^2 + y^2 + z^2$. Then $\nabla w = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$, and $\nabla w(0,0,3) = 6\hat{k}$. Hence, the equation for the tangent plane is $6(z-3) = 0 \Rightarrow z = 3$.

8. Find the equation of the tangent plane to the surface $x^2 + y^2 - z^2 = 0$ at the point $P = (3,4,5)$.

Let $w = x^2 + y^2 - z^2$. Then $\nabla w = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$, and $\nabla w(3,4,5) = 6\hat{i} + 8\hat{j} - 10\hat{k}$. Hence, the equation for the tangent plane is

$6(x-3) + 8(y-4) - 10(z-5) = 0 \Rightarrow 6x - 18 + 8y - 32 - 10z + 50 = 0$

$\Rightarrow 6z + 8y - 10z = 0 \Rightarrow 3x + 4y - 5z = 0 \Rightarrow z = \frac{3}{5}x + \frac{4}{5}y$ ..