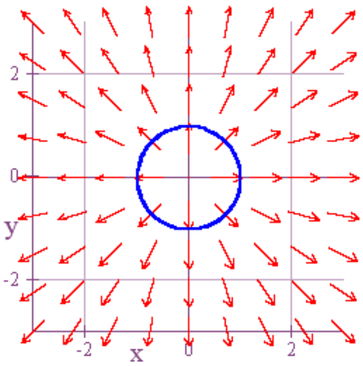
# GREENS THEOREM AND STOKES THEOREM - ANSWERS

Use Green's Theorem (which in 2-dimensions is the same as Stoke's Theorem),

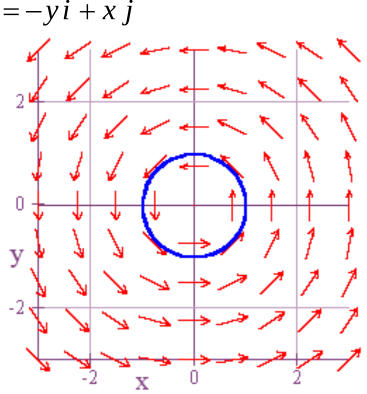
$Circulation = \int_C \vec{F} \bullet T\, ds = \int_C \vec{F} \bullet d\vec{r} = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R \left( \nabla \times \vec{F} \right) \bullet \hat{k}\, dA$, to measure the circulation around the boundary of the unit circle (oriented counterclockwise) caused by each of the following vector fields.

1. $\vec{F} = x\hat{i} + y\hat{j}$



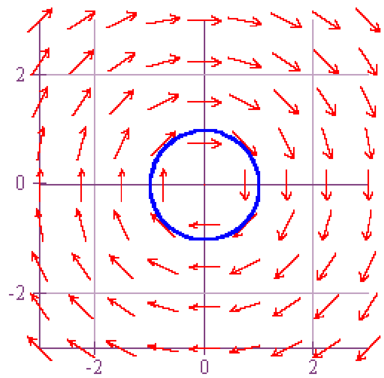

$$\iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R (0-0)\, dA = 0$$

2. $\vec{F} = -y\hat{i} + x\hat{j}$



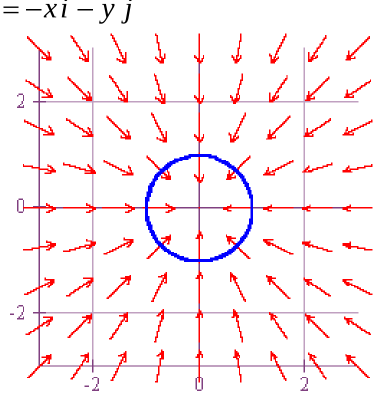

$$\iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R (1-(-1))\, dA = 2\pi$$

3. $\vec{F} = y\hat{i} - x\hat{j}$



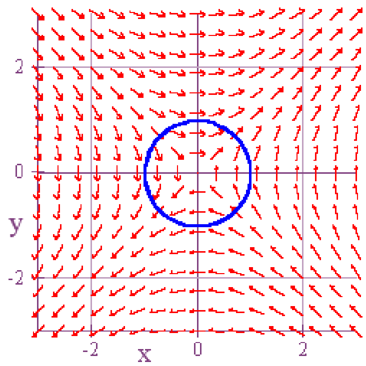

$$\iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R (-1-1)\, dA = -2\pi$$

4. $\vec{F} = -x\hat{i} - y\hat{j}$



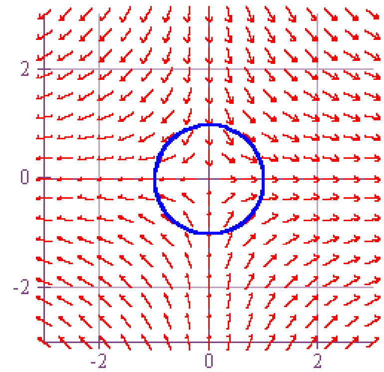

$$\iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R (0-0)\, dA = 0$$

5. $\vec{F} = y\hat{i} + x\hat{j}$




$$\iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R (1-1)\, dA = 0$$

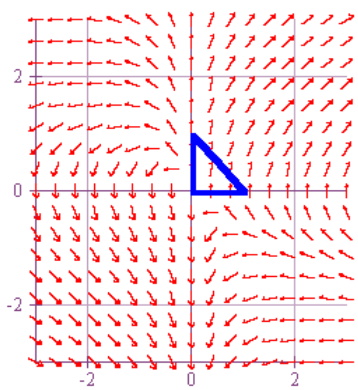
6. $\vec{F} = 4x\hat{i} - 3y\hat{j}$




$$\iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R (0-0)\, dA = 0$$

7. $\vec{F} = xy\hat{i} + (x+y)\hat{j}$

$R$ is a triangular region defined by $0 \le x \le 1$ and $0 \le y \le -x+1$, and $R$ is oriented counterclockwise. Use Green's Theorem to find the circulation around the curve created by the vector field $\vec{F}$.



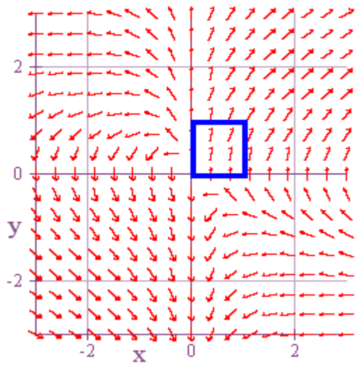

$$\int_C \vec{F} \bullet T\, ds = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^1 \int_0^{1-x} (1-x)\, dy dx = \int_0^1 (1-x) y \Big|_0^{1-x} dx = \int_0^1 (1-x)^2\, dx$$

$$= \int_0^1 (x^2 - 2x + 1)\, dx = \frac{x^3}{3} - x^2 + x \Big|_0^1 = \frac{1}{3}$$

8. $\vec{F} = xy\hat{i} + (x+y)\hat{j}$

$R$ is a square region defined by $0 \le x \le 1$ and $0 \le y \le 1$, and $R$ is oriented counterclockwise.

Use Green's Theorem to find the circulation around the curve created by the vector field $\vec{F}$.



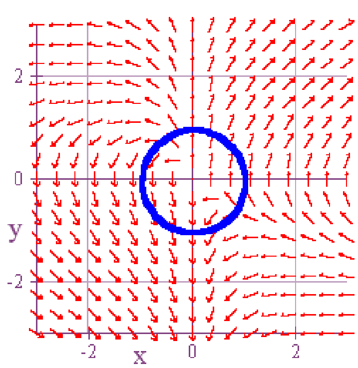

$$\int_C \vec{F} \bullet T\, ds = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^1 \int_0^1 (1-x)\, dydx = \int_0^1 (1-x)y \Big|_0^1 dx = \int_0^1 (1-x)\, dx$$

$$= x - \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}$$

9. $\vec{F} = xy\hat{i} + (x+y)\hat{j}$

$R$ is the unit circle, and $R$ is oriented counterclockwise.

Use Green's Theorem to find the circulation around the curve created by the vector field $\vec{F}$.




$$\int_C \vec{F} \bullet T\, ds = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R (1-x)\, dA = \int_0^{2\pi} \int_0^1 (1 - r\cos\theta) r\, dr d\theta = \int_0^{2\pi} \int_0^1 (r - r^2 \cos\theta)\, dr d\theta$$

$$= \int_0^{2\pi} \left( \frac{r^2}{2} - \frac{r^3}{3}\cos\theta \right) \Big|_0^1 d\theta = \int_0^{2\pi} \left( \frac{1}{2} - \frac{1}{3}\cos\theta \right) d\theta = \frac{\theta}{2} - \frac{\sin\theta}{3} \Big|_0^{2\pi} = \pi$$

10. Evaluate $\int_C ye^{-x}dx + (x^2/2 - e^{-x})dy$ where $C$ is the circle of radius 1 with center at (2,0), oriented counterclockwise. (HINT: Use a change a variables to move the center to the origin, and then integrate using polar coordinates.)

By Green's Theorem, $\int_C ye^{-x}dx + (x^2/2 - e^{-x})dy = \iint_R x\,dA = \iint_R x\,dydx$. To shift the center from (2,0) to the origin, let $u = x-2, v = y$. Then $x = u+2, y = v$ and $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right\| = 1$. Hence,

$$\iint_R x\,dydx = \iint_{unit\,circle} (u+2)dvdu = \iint_{unit\,circle} u\,dvdu + \iint_{unit\,circle} 2\,dvdu = \int_0^{2\pi}\int_0^1 r\cos\theta\, rdrd\theta + 2\iint_{unit\,cirle} dvdu$$

$$= \int_0^{2\pi} \frac{1}{3}\cos\theta\, d\theta + 2\pi = \left(\frac{1}{3}\sin 2\pi - \frac{1}{3}\sin 0\right) + 2\pi = 2\pi.$$