# INDEPENDENCE OF PATH - ANSWERS

For each vector field below in problems 1 through 5, evaluate $\int_C \vec{F} \bullet d\vec{r}$ where $C$ is any path from $(0,0)$ to $(1,1)$.

1. $\vec{F} = x\hat{i} + y\hat{j}$

$$\int_C \vec{F} \bullet d\vec{r} = \int_{(0,0)}^{(1,1)} \vec{F} \bullet d\vec{r} = \frac{x^2}{2} + \frac{y^2}{2}\Bigg|_{(0,0)}^{(1,1)} = \frac{1}{2} + \frac{1}{2} = 1$$

2. $\vec{F} = y\hat{i} + x\hat{j}$

$$\int_C \vec{F} \bullet d\vec{r} = \int_{(0,0)}^{(1,1)} \vec{F} \bullet d\vec{r} = xy\Big|_{(0,0)}^{(1,1)} = (1)(1) = 1$$

3. $\vec{F} = \cos(x)\hat{i} + \sin(y)\hat{j}$

$$\int_C \vec{F} \bullet d\vec{r} = \int_{(0,0)}^{(1,1)} \vec{F} \bullet d\vec{r} = \left[\sin x - \cos y\right]\Big|_{(0,0)}^{(1,1)} = \sin(1) - \cos(1) + 1$$

4. $\vec{F} = (e^x + y^2)\hat{i} + (\cos y + 2xy)\hat{j}$

$$\int_C \vec{F} \bullet d\vec{r} = \int_{(0,0)}^{(1,1)} \vec{F} \bullet d\vec{r} = e^x + xy^2 + \sin y\Big|_{(0,0)}^{(1,1)} = e + 1 + \sin(1) - 1 - 0 - 0 = e + \sin(1)$$

5. $\vec{F} = (3xy^2 + 5)\hat{i} + (3 + 3x^2y)\hat{j}$

$$\int_C \vec{F} \bullet d\vec{r} = \int_{(0,0)}^{(1,1)} \vec{F} \bullet d\vec{r} = \frac{3x^2y^2}{2} + 5x + 3y\Bigg|_{(0,0)}^{(1,1)} = \frac{3}{2} + 5 + 3 = \frac{19}{2}$$

6. Suppose $\vec{F}$ is a conservative vector field (i.e. a gradient field) with potential *f*, and suppose $C$ is a path from $(a,b)$ to $(c,d)$ and that $-C$ is the same path traversed in the opposite direction. Explain why $\int_{C+(-C)} \vec{F} \bullet d\vec{r} = \int_C \vec{F} \bullet d\vec{r} + \int_{-C} \vec{F} \bullet d\vec{r} = 0$.

   Clearly, $\int_{C+(-C)} \vec{F} \bullet d\vec{r} = \int_C \vec{F} \bullet d\vec{r} + \int_{-C} \vec{F} \bullet d\vec{r} = f(c,d) - f(a,b) + f(a,b) - f(c,d) = 0$.

7. If $\vec{F}$ is a gradient field and $\int_{(0,0)}^{(1,0)} \vec{F} \bullet d\vec{r} = 5$, $\int_{(1,0)}^{(1,1)} \vec{F} \bullet d\vec{r} = 10$, and $\int_{(0,1)}^{(1,1)} \vec{F} \bullet d\vec{r} = 20$, find $\int_{(0,1)}^{(0,0)} \vec{F} \bullet d\vec{r}$.

   $$\int_{(0,1)}^{(0,0)} \vec{F} \bullet d\vec{r} = \int_{(0,1)}^{(1,1)} \vec{F} \bullet d\vec{r} + \int_{(1,1)}^{(1,0)} \vec{F} \bullet d\vec{r} + \int_{(1,0)}^{(0,0)} \vec{F} \bullet d\vec{r} = 20 - 10 - 5 = 5.$$

8. Isaac Newton discovered that *Force = mass time acceleration*, or as we express it mathematically, $F = ma = m\frac{d^2y}{dx^2}$. In particular, the force due to gravity is represented by the vector field $\vec{F} = -mg\,\hat{k}$. Thus, if you are moving upwards, the force you exert against gravity can be represented by the vector field $\vec{F}_1 = mg\,\hat{k}$. For the sake of simplicity, assume that units of feet, pounds, and seconds are being used.
   a. Find a potential function *f* for $\vec{F}_1 = mg\,\hat{k}$.
   b. If you ascend 1000 feet up a winding staircase, how much work do you do?
   a. If you walk to the top of the Empire State Building and back down again, how much work do you do?

   a. $f = mgz$.
   b. $\int_C \vec{F}_1 \bullet d\vec{r} = f(0,0,1000) - f(0,0,0) = 1000mg$.
   c. $\int_C \vec{F}_1 \bullet d\vec{r} = f(0,0,0) - f(0,0,0) = 0$

9. Given that $\vec{F} = \cos x \sin y \sin z\,\hat{i} + \sin x \cos y \sin z\,\hat{j} + \sin x \sin y \cos z\,\hat{k}$ is a gradient field, find the value of $\int_C \vec{F} \bullet d\vec{r}$ along any path from $(0,0,0)$ to $\left(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right)$.

$$f(x,y,z) = \int \cos x \sin y \sin z\, dx = \sin x \sin y \sin z + g(y,z)$$
$$\sin x \cos y \sin z = f_y = \sin x \cos y \sin z + g_y \Rightarrow f(x,y,z) = \sin x \sin y \sin z + h(z)$$
$$\sin x \sin y \cos z = f_z = \sin x \sin y \cos z + h_z \Rightarrow f(x,y,z) = \sin x \sin y \sin z$$

Therefore, $\int_C \vec{F} \bullet d\vec{r} = f\left(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right) - f(0,0,0) = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot 1 - 0 \cdot 0 \cdot 0 = \frac{\sqrt{3}}{4}$