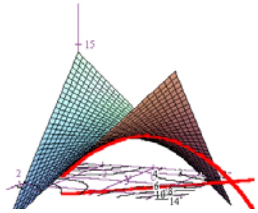
LAGRANGE MULTIPLIERS - ANSWERS

Use the method of Lagrange multipliers to solve the following problems.

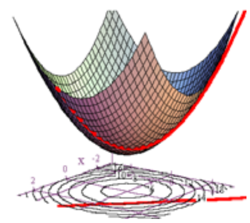


1. Find the coordinates of the maximum point on the graph of $z = xy + 5$ subject to the constraint $x + y = 2$.

$$x + y = 2$$
$$g = x + y$$

$$z_x = y \quad g_x = 1$$
$$z_y = x \quad g_y = 1$$

$$\begin{matrix} z_x = \lambda g_x \\ z_y = \lambda g_y \end{matrix} \Rightarrow \begin{matrix} y = \lambda \\ x = \lambda \end{matrix} \Rightarrow \lambda + \lambda = 2 \Rightarrow \lambda = 1 \Rightarrow \begin{matrix} x = 1 \\ y = 1 \\ z = 6 \end{matrix}$$



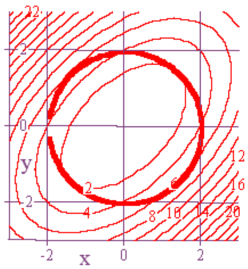
2. Find the coordinates of the minimum point on the graph of $z = x^2 + y^2 + 5$ subject to the constraint $x + y = 2$.

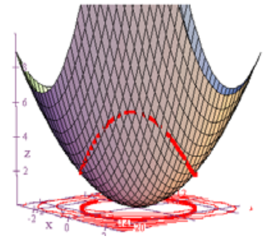
$$x + y = 2$$
$$g = x + y$$

$$z_x = 2x \quad g_x = 1$$
$$z_y = 2y \quad g_y = 1$$

$$\begin{matrix} z_x = \lambda g_x \\ z_y = \lambda g_y \end{matrix} \Rightarrow \begin{matrix} 2x = \lambda \\ 2y = \lambda \end{matrix} \Rightarrow \begin{matrix} x = \lambda/2 \\ y = \lambda/2 \end{matrix} \Rightarrow \frac{\lambda}{2} + \frac{\lambda}{2} = 2 \Rightarrow \lambda = 2 \Rightarrow \begin{matrix} x = 1 \\ y = 1 \\ z = 7 \end{matrix}$$

3. Find the coordinates of the extreme points on the graph of $z = x^2 - xy + y^2$ subject to the constraint $x^2 + y^2 = 4$.



$$x^2 + y^2 = 4$$
$$g = x^2 + y^2$$

$$z_x = 2x - y \quad g_x = 2x$$
$$z_y = 2y - x \quad g_y = 2y$$

$$\begin{matrix} z_x = \lambda g_x \\ z_y = \lambda g_y \end{matrix} \Rightarrow \begin{matrix} 2x - y = 2\lambda x \\ 2y - x = 2\lambda y \end{matrix} \Rightarrow \frac{2x - y}{2y - x} = \frac{2\lambda x}{2\lambda y} \Rightarrow 2xy - x^2 = 2xy - y^2$$
$$\Rightarrow x^2 = y^2 \Rightarrow x^2 + x^2 = 4 \Rightarrow x = \pm\sqrt{2}$$
$$\Rightarrow \left(-\sqrt{2}, -\sqrt{2}\right) or \left(\sqrt{2}, \sqrt{2}\right) or \left(-\sqrt{2}, \sqrt{2}\right) or \left(\sqrt{2}, -\sqrt{2}\right)$$

$\left(-\sqrt{2}, -\sqrt{2}, 2\right)$ is a minimum point

$\left(\sqrt{2}, \sqrt{2}, 2\right)$ is a minimum point

$\left(-\sqrt{2}, \sqrt{2}, 6\right)$ is a maximum point

$\left(\sqrt{2}, -\sqrt{2}, 6\right)$ is a maximum point

4. Let $w = xyz$ for $x > 0$, $y > 0$, and $z > 0$. Find the maximum value of *w* subject to the constraint $x + y + z = 48$.

$$x + y + z = 48$$
$$g = x + y + z$$

$$w_x = yz \quad g_x = 1$$
$$w_y = xz \quad g_y = 1$$
$$w_z = xy \quad g_z = 1$$

$$\begin{matrix} w_x = \lambda g_x \\ w_y = \lambda g_y \\ w_z = \lambda g_z \end{matrix} \Rightarrow \begin{matrix} yz = \lambda \\ xz = \lambda \\ xy = \lambda \end{matrix} \Rightarrow \begin{matrix} \frac{yz}{xz} = \frac{\lambda}{\lambda} = 1 \\ \frac{xz}{xy} = \frac{\lambda}{\lambda} = 1 \end{matrix} \Rightarrow \begin{matrix} y = x \\ z = y \\ x = z \end{matrix} \Rightarrow \begin{matrix} x = \frac{48}{3} = 16 \\ y = \frac{48}{3} = 16 \\ z = \frac{48}{3} = 16 \\ w = 16^3 = 4,096 \end{matrix}$$

NOTE: We've assumed in our work that $\lambda \neq 0$. Why is this reasonable? What happens to the values of our variables if $\lambda = 0$?

5. A manufacturer has an order for 1000 ultra-deluxe time machines with built-in MP3 player. Suppose the units are manufactured in two different locations with *x* representing the number of units produced in one location and *y* the number of units in the other. If the total cost of production is given by $z = C(x, y) = x^2 + 10x + 0.50y^2 + 12y - 10,000 \; dollars$, find the values of *x* and *y* that will minimize the costs and find the minimum cost.

$$x + y = 1000 \qquad z_x = 2x + 10 \quad g_x = 1$$
$$g = x + y \qquad z_y = y + 12 \quad g_y = 1$$

$$\begin{matrix} z_x = \lambda g_x \\ z_y = \lambda g_y \end{matrix} \Rightarrow \begin{matrix} 2x + 10 = \lambda \\ y + 12 = \lambda \end{matrix} \Rightarrow \begin{matrix} x = \frac{\lambda - 10}{2} \\ y = \lambda - 12 \end{matrix} \Rightarrow \frac{\lambda - 10}{2} + \lambda - 12 = 1000$$

$$\Rightarrow \lambda - 10 + 2\lambda - 24 = 2000 \Rightarrow 3\lambda = 2034 \Rightarrow \lambda = 678$$

$$x = \frac{678 - 10}{2} = 334$$
$$y = 678 - 12 = 666$$
$$z = 334,666$$

6. Find the points on the circle $x^2 + y^2 = 100$ that are closest to and farthest from the point $(2,3)$.

We can simplify the derivative process by looking for maximum and minimum values for the square of the distance from $(2,3)$ rather than the distance from $(2,3)$ itself. Thus, let $z = f(x,y) = (x-2)^2 + (y-3)^2$ and let $g = g(x,y) = x^2 + y^2$. Then $z_x = 2(x-2) = 2x-4$, $z_y = 2(y-3) = 2y-6$, $g_x = 2x$, and $g_y = 2y$. Thus,

$$\begin{array}{rcl} 2x-4 & = & 2\lambda x \\ 2y-6 & = & 2\lambda y \end{array} \Rightarrow \begin{array}{rcl} \dfrac{2x-4}{x} & = & 2\lambda \\ \dfrac{2y-6}{y} & = & 2\lambda \end{array} \Rightarrow \frac{2x-4}{x} = \frac{2y-6}{y}$$

$\Rightarrow 2xy - 4y = 2xy - 6x \Rightarrow -4y = -6x \Rightarrow y = \dfrac{3}{2}x$. Therefore, $x^2 + y^2 = 100$

$\Rightarrow x^2 + \dfrac{9}{4}x^2 = \dfrac{13}{4}x^2 = 100 \Rightarrow x^2 = \dfrac{400}{13} \Rightarrow x = \dfrac{20}{\sqrt{13}}, y = \dfrac{30}{\sqrt{13}}$ or $x = -\dfrac{20}{\sqrt{13}}, y = -\dfrac{30}{\sqrt{13}}$.

The corresponding function values are $f\left(\dfrac{20}{\sqrt{13}}, \dfrac{30}{\sqrt{13}}\right) \approx 40.9$ and $f\left(-\dfrac{20}{\sqrt{13}}, -\dfrac{30}{\sqrt{13}}\right) \approx 185.1$. Therefore, the point on the circle that is closest to $(2,3)$ is $\left(\dfrac{20}{\sqrt{13}}, \dfrac{30}{\sqrt{13}}\right)$, and the point on the circle farthest from $(2,3)$ is $\left(-\dfrac{20}{\sqrt{13}}, -\dfrac{30}{\sqrt{13}}\right)$.

7. Find the area of the largest rectangle that can be inscribed inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Again, we can make things easier by this time taking advantage of the symmetry of the ellipse. In other words, solve the problem only for the portion of the ellipse in the first quadrant where both *x* and *y* are positive, and then multiply the resulting area by 4. Thus, let $z = A(x,y) = xy$ and let $g = g(x,y) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$. Then,

$$\begin{array}{rcl} z_x & = & y \\ z_y & = & x \end{array} \text{ and } \begin{array}{rcl} g_x & = & \dfrac{2x}{a^2} \\ g_y & = & \dfrac{2y}{b^2} \end{array} \Rightarrow \begin{array}{rcl} y & = & \dfrac{2\lambda x}{a^2} \\ x & = & \dfrac{2\lambda y}{b^2} \end{array} \Rightarrow \frac{y}{x} = \frac{\frac{2\lambda x}{a^2}}{\frac{2\lambda y}{b^2}} = \frac{b^2 x}{a^2 y} \Rightarrow a^2 y^2 = b^2 x^2$$

$$\Rightarrow y^2 = \frac{b^2}{a^2}x^2 \Rightarrow y = \frac{b}{a}x. \text{ Hence, } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{\frac{b^2 x^2}{a^2}}{b^2} = 1 \Rightarrow \frac{2x^2}{a^2} = 1 \Rightarrow x^2 = \frac{a^2}{2}$$

$\Rightarrow x = \dfrac{a}{\sqrt{2}}$ and $y = \dfrac{b}{\sqrt{2}}$. Thus, the maximum area is $4xy = 4 \cdot \dfrac{a}{\sqrt{2}} \cdot \dfrac{b}{\sqrt{2}} = 2ab$.

8. Find the volume of the largest rectangular box that can be inscribed inside the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Again, we can take advantage of symmetry by solving the problem for the first octant where all variables are nonnegative and then multiplying this result by 8. Thus, let $V = V(x,y,z) = xyz$, and let $g = g(x,y,z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$. Then,

$V_x = yz$, $V_y = xz$, $V_z = xy$, $g_x = \dfrac{2x}{a^2}$, $g_y = \dfrac{2y}{b^2}$, and $g_z = \dfrac{2z}{c^2}$. Hence,

$$\begin{aligned} yz &= \frac{2\lambda x}{a^2} \\ xz &= \frac{2\lambda y}{b^2} \Rightarrow \frac{yz}{xz} = \frac{\frac{2\lambda x}{a^2}}{\frac{2\lambda y}{b^2}} \Rightarrow \frac{y}{x} = \frac{b^2 x}{a^2 y} \Rightarrow y^2 = \frac{b^2 x^2}{a^2} \Rightarrow y = \frac{bx}{a}. \text{ Additionally,} \\ xy &= \frac{2\lambda z}{c^2} \end{aligned}$$

$\dfrac{yz}{xy} = \dfrac{\frac{2\lambda x}{a^2}}{\frac{2\lambda z}{c^2}} \Rightarrow \dfrac{z}{x} = \dfrac{c^2 x}{a^2 z} \Rightarrow z^2 = \dfrac{c^2 x^2}{a^2} \Rightarrow z = \dfrac{cx}{a}$. Thus, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

$\dfrac{x^2}{a^2} + \dfrac{x^2}{a^2} + \dfrac{x^2}{a^2} = \dfrac{3x^2}{a^2} = 1 \Rightarrow x = \dfrac{a}{\sqrt{3}} \Rightarrow y = \dfrac{b}{\sqrt{3}}$ & $z = \dfrac{c}{\sqrt{3}}$. Therefore, the maximum volume is $8xyz = \dfrac{8abc}{3\sqrt{3}}$.

9. Use Lagrange multipliers to find the point in the plane $2x + y - z = -5$ that is closest to the origin. (HINT: Minimize the square of the distance from the origin. You will get the same answer, but you won't have to mess with derivatives of square roots.)

Let $f = x^2 + y^2 + z^2$ and let $g = 2x + y - z$.

$$\begin{matrix} f_x = 2x & & g_x = 2 & & 2x = 2\lambda & & x = \lambda \\ f_y = 2y & \& & g_y = 1 & \Rightarrow & 2y = \lambda & \Rightarrow & y = \frac{\lambda}{2} \\ f_z = 2z & & g_z = -1 & & 2z = -\lambda & & z = -\frac{\lambda}{2} \end{matrix}$$

Thus, $2x + y - z = -5 \Rightarrow 2\lambda + \frac{\lambda}{2} + \frac{\lambda}{2} = -5 \Rightarrow \lambda = -\frac{5}{3} \Rightarrow \begin{matrix} x = -\frac{5}{3} \\ y = -\frac{5}{6} \\ z = \frac{5}{6} \end{matrix}$

10. A company operates two plants which manufacture the same item. Suppose that the total cost involved in producing quantities $q_1$ and $q_2$ at the two plants is $C = 2q_1^2 + q_1 q_2 + q_2^2 + 1000$. Suppose also that the company's objective is to produce a total quantity of $q_1 + q_2 = 100$ units. Find levels of production, $q_1$ and $q_2$, that will minimize the cost.

Let $C = 2q_1^2 + q_1 q_2 + q_2^2 + 1000$ and let $g = q_1 + q_2$. Then,

$$\begin{matrix} C_{q_1} = \lambda g_{q_1} \\ C_{q_2} = \lambda g_{q_2} \end{matrix} \Rightarrow \begin{matrix} 4q_1 + q_2 = \lambda \\ q_1 + 2q_2 = \lambda \end{matrix} \Rightarrow 4q_1 + q_2 = q_1 + 2q_2 \Rightarrow q_2 = 3q_1$$

$$q_1 + q_2 = 100 \Rightarrow q_1 + 3q_2 = 100 \Rightarrow \begin{matrix} q_1 = 25 \\ q_2 = 75 \end{matrix}.$$

Therefore, the company should produce 25 items at the first plant and 75 at the second plant in order to minimize costs.

$q_1 + q_2$