LIMITS – ANSWERS

Either find the indicated limit, or show that the limit fails to exist by evaluating it along two different paths and arriving at two different results.

1. $\lim_{(x,y)\to(0,0)} \cos(xy)$

   Since the function is continuous at (0,0), $\lim_{(x,y)\to(0,0)} \cos(xy) = \cos(0\cdot 0) = \cos(0) = 1$.

2. $\lim_{(x,y)\to(0,0)} e^{xy}$

   Since the function is continuous at (0,0), $\lim_{(x,y)\to(0,0)} e^{xy} = e^{0\cdot 0} = e^0 = 1$

3. $\lim_{(x,y)\to(0,0)} \frac{3x^2 - 2y^2}{x^2 + y^2}$

   If we take the limit along the path $y = 0$, then

   $$\lim_{(x,y)\to(0,0)} \frac{3x^2 - 2y^2}{x^2 + y^2} = \lim_{(x,0)\to(0,0)} \frac{3x^2 - 2\cdot 0^2}{x^2 + 0^2} = \lim_{x\to 0} \frac{3x^2}{x^2} = \lim_{x\to 0} 3 = 3.$$

   On the other hand, if we take the limit along the path $x = 0$, then

   $$\lim_{(x,y)\to(0,0)} \frac{3x^2 - 2y^2}{x^2 + y^2} = \lim_{(0,y)\to(0,0)} \frac{3\cdot 0^2 - 2y^2}{0^2 + y^2} = \lim_{y\to 0} \frac{-2y^2}{y^2} = \lim_{x\to 0}(-2) = -2.$$

   Since the two limit values are different, the general limit does not exist.

4. $\lim_{(x,y)\to(1,-1)} \frac{x^2 - 2xy + y^2}{x - y}$

   $$\lim_{(x,y)\to(1,-1)} \frac{x^2 - 2xy + y^2}{x - y} = \lim_{(x,y)\to(1,-1)} \frac{(x-y)^2}{x - y} = \lim_{(x,y)\to(1,-1)} (x - y) = 1 - (-1) = 2.$$

5. $\lim_{(x,y)\to(0,0)} \frac{x^2 - 2xy + y^2}{x - y}$

   $$\lim_{(x,y)\to(0,0)} \frac{x^2 - 2xy + y^2}{x - y} = \lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x - y} = \lim_{(x,y)\to(0,0)} (x - y) = 0 - 0 = 0.$$

6. $\lim_{(x,y)\to(1,1)} \frac{x^2 - y^2}{x - y}$

$$\lim_{(x,y)\to(1,1)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y)\to(1,1)} \frac{(x+y)(x-y)}{x-y} \lim_{(x,y)\to(1,1)} (x+y) = 1+1 = 2$$

7. $\lim_{(x,y)\to(0,0)} \frac{x}{y}$

If we take the limit along the path $y = x$, then

$$\lim_{(x,y)\to(0,0)} \frac{x}{y} = \lim_{(x,x)\to(0,0)} \frac{x}{x} = \lim_{x\to 0}(1) = 1.$$

On the other hand, if we take the limit along the path $y = -x$, then

$$\lim_{(x,y)\to(0,0)} \frac{x}{y} = \lim_{(x,x)\to(0,0)} \frac{x}{-x} = \lim_{x\to 0}(-1) = -1.$$

Since the two limit values are different, the general limit does not exist.

8. $\lim_{(x,y)\to(0,0)} \frac{x^4 y^2}{x^8 + y^4}$

If we take the limit along the path $y = 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y^2}{x^8 + y^4} = \lim_{(x,0)\to(0,0)} \frac{x^4 \cdot 0^2}{x^8 + 0^4} = \lim_{x\to 0} \frac{0}{x^8} = 0.$$

On the other hand, if we take the limit along the path $y = x^2$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y^2}{x^8 + y^4} = \lim_{(x,0)\to(0,0)} \frac{x^4 \cdot (x^2)^2}{x^8 + (x^2)^4} = \lim_{x\to 0} \frac{x^4 x^4}{x^8 + x^8} = \lim_{x\to 0} \frac{x^8}{2x^8} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since the two limit values are different, the general limit does not exist.