LINE INTEGRALS - ANSWERS

1. Evaluate $\int_C xy\,ds$ where C is the unit circle traversed once in the counterclockwise direction.

$$\int_C xy\,ds = \int_a^b x(t)y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\begin{array}{r} x = \cos t \\ y = \sin t \Rightarrow \\ 0 \le t \le 2\pi \end{array} \begin{array}{l} \dfrac{dx}{dt} = -\sin t \\ \dfrac{dy}{dt} = \cos t \end{array} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1$$

$$\int_C xy\,ds = \int_a^b x(t)y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi} \cos t \sin t\,dt = -\int_0^{2\pi} \cos t\left(-\sin t\right)dt = -\int_1^1 u\,du = 0$$

$$\begin{array}{r} u = \cos t \\ du = -\sin t\,dt \\ 1 \le u \le 1 \end{array}$$

2. Evaluate $\int_C xy\,dx$ where C is the unit circle traversed once in the counterclockwise direction.

$$\int_C xy\,dx = \int_a^b x(t)y(t)\frac{dx}{dt}\,dt$$

$$\begin{array}{r} x = \cos t \\ y = \sin t \Rightarrow \\ 0 \le t \le 2\pi \end{array} \frac{dx}{dt} = -\sin t$$

$$\int_C xy\,dx = \int_a^b x(t)y(t)\frac{dx}{dt}\,dt = \int_0^{2\pi} \cos t \sin t(-\sin t)\,dt = \int_0^{2\pi} -\sin^2 t\cos t\,dt = \int_0^0 -u^2\,du = 0$$

$$\begin{array}{r} u = \sin t \\ du = \cos t\,dt \\ 0 \le u \le 0 \end{array}$$

3. Evaluate $\int_C xy\,dy$ where C is the unit circle traversed once in the counterclockwise direction.

$$\int_C xy\,dy = \int_a^b x(t)y(t)\frac{dy}{dt}\,dt$$

$$\begin{aligned} x &= \cos t \\ y &= \sin t \Rightarrow \frac{dy}{dt} = \cos t \\ 0 &\le t \le 2\pi \end{aligned}$$

$$\int_C xy\,dy = \int_a^b x(t)y(t)\frac{dy}{dt}\,dt = \int_0^{2\pi} \cos t \sin t \cos t\,dt = \int_0^{2\pi} \cos^2 t \sin t\,dt = \int_1^1 -u^2\,du = 0$$

$$\begin{aligned} u &= \cos t \\ du &= -\sin t\,dt \\ 1 &\le u \le 1 \end{aligned}$$

4. Evaluate $\int_C xy\,ds$ where C is the straight line from $(0,0)$ to $(2,4)$.

$$\int_C xy\,ds = \int_a^b x(t)y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\begin{aligned} x &= 2t \\ y &= 4t \Rightarrow \\ 0 &\le t \le 1 \end{aligned} \quad \begin{aligned} \frac{dx}{dt} &= 2 \\ \frac{dy}{dt} &= 4 \end{aligned} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

$$\int_C xy\,ds = \int_a^b x(t)y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 8t^2 \cdot 2\sqrt{5}\,dt = 16\sqrt{5}\left.\frac{t^3}{3}\right|_0^1 = \frac{16\sqrt{5}}{3}$$

5. Evaluate $\int_C xy\,dx$ where C is the straight line from $(0,0)$ to $(2,4)$.

$$\int_C xy\,dx = \int_a^b x(t)y(t)\frac{dx}{dt}\,dt$$

$$\begin{aligned} x &= 2t \\ y &= 4t \Rightarrow \frac{dx}{dt} = 2 \\ 0 \le t &\le 1 \end{aligned}$$

$$\int_C xy\,dx = \int_a^b x(t)y(t)\frac{dx}{dt}\,dt = \int_0^1 8t^2 \cdot 2dt = \left.\frac{16t^3}{3}\right|_0^1 = \frac{16}{3}$$

6. Evaluate $\int_C xy\,dy$ where C is the straight line from $(0,0)$ to $(2,4)$.

$$\int_C xy\,dy = \int_a^b x(t)y(t)\frac{dy}{dt}\,dt$$

$$\begin{aligned} x &= 2t \\ y &= 4t \Rightarrow \frac{dy}{dt} = 4 \\ 0 \le t &\le 1 \end{aligned}$$

$$\int_C xy\,dy = \int_a^b x(t)y(t)\frac{dy}{dt}\,dt = \int_0^1 8t^2 \cdot 4dt = \left.\frac{32t^3}{3}\right|_0^1 = \frac{32}{3}$$

7. Evaluate $\int_C x\,ds$ where C is the curve $y = x^2$ where $0 \le x \le 1$.

$$\int_C x\,ds = \int_a^b x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\begin{array}{c} x = t \\ y = t^2 \\ 0 \le t \le 1 \end{array} \Rightarrow \begin{array}{c} \dfrac{dx}{dt} = 1 \\ \dfrac{dy}{dt} = 2t \end{array} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1+4t^2}$$

$$\int_C x\,ds = \int_a^b x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 t\sqrt{1+4t^2}\,dt = \frac{1}{8}\int_1^5 u^{1/2}\,du = \frac{1}{12}\left(5\sqrt{5} - 1\right)$$

$$\begin{array}{c} u = 1 + 4t^2 \\ du = 8t\,dt \\ 1 \le u \le 5 \end{array}$$

8. Evaluate $\int_C x\,dx$ where C is the curve $y = x^2$ where $0 \le x \le 1$.

$$\int_C x\,dx = \int_a^b x(t)\frac{dx}{dt}\,dt$$

$$\begin{array}{c} x = t \\ y = t^2 \\ 0 \le t \le 1 \end{array} \Rightarrow \frac{dx}{dt} = 1$$

$$\int_C x\,dx = \int_a^b x(t)\frac{dx}{dt}\,dt = \int_0^1 t\,dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2}$$

9. Evaluate $\int_C x\,dy$ where C is the curve $y = x^2$ where $0 \le x \le 1$.

$$\int_C x\,dy = \int_a^b x(t)\frac{dy}{dt}\,dt$$

$$\begin{aligned} x &= t \\ y &= t^2 \Rightarrow \frac{dy}{dt} = 2t \\ 0 &\le t \le 1 \end{aligned}$$

$$\int_C x\,dy = \int_a^b x(t)\frac{dy}{dt}\,dt = \int_0^1 2t^2\,dt = \left.\frac{2t^3}{3}\right|_0^1 = \frac{2}{3}$$

10. Suppose a wire is shaped into a path corresponding to $x = \cos t$, $y = \sin t$, and $z = t/5$ for $0 \le t \le 6\pi$, and suppose also that the density of the wire in terms of mass per unit of length changes with elevation according to the function $w(x, y, z) = z$. Then find the mass of the wire.

We can express the path by the vector-valued function $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + \frac{t}{5}\hat{k}$, and the then mass is equal to

$$\int_C w(x,y,z)\,ds = \int_C z\,ds = \int_0^{6\pi} \frac{t}{5}\cdot\left\|r'(t)\right\|dt = \int_0^{6\pi} \frac{t}{5}\cdot\sqrt{(-\sin t)^2 + (\cos t)^2 + (1/5)^2}\,dt$$

$$= \int_0^{6\pi} \frac{t}{5}\cdot\frac{\sqrt{26}}{5}\,dt = \left.\frac{t^2\sqrt{26}}{50}\right|_0^{6\pi} = \frac{36\pi^2\sqrt{26}}{50} = \frac{18\pi^2\sqrt{26}}{25} \approx 36.23$$