# CHAPTER 12

# Motion Along a Curve

## 12.1 The Position Vector

This chapter is about "vector functions." The vector $2\mathbf{i}+4\mathbf{j}+8\mathbf{k}$ is constant. The vector $\mathbf{R}(t)=t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$ is moving. It is a function of the parameter $t$, which often represents time. At each time $t$, the position vector $\mathbf{R}(t)$ locates the moving body:

$$\textbf{\textit{position vector}} = \mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}. \tag{1}$$

Our example has $x=t$, $y=t^2$, $z=t^3$. As $t$ varies, these points trace out a ***curve in space***. The parameter $t$ tells when the body passes each point on the curve. The constant vector $2\mathbf{i}+4\mathbf{j}+8\mathbf{k}$ is the position vector $\mathbf{R}(2)$ at the instant $t=2$.

What are the questions to be asked? Every student of calculus knows the first question: *Find the derivative*. If something moves, the Navy salutes it and we differentiate it. At each instant, the body moving along the curve has a speed and a direction. This information is contained in another vector function—the velocity vector $\mathbf{v}(t)$ which is the derivative of $\mathbf{R}(t)$:

$$\mathbf{v}(t)=\frac{d\mathbf{R}}{dt}=\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}+\frac{dz}{dt}\mathbf{k}. \tag{2}$$

Since $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are fixed vectors, their derivatives are zero. In polar coordinates $\mathbf{i}$ and $\mathbf{j}$ are replaced by moving vectors. Then the velocity $\mathbf{v}$ has more terms from the product rule (Section 12.4).

Two important cases are uniform motion ***along a line and around a circle***. We study those motions in detail ($\mathbf{v}$ = constant on line, $\mathbf{v}$ = tangent to circle). This section also finds the speed and distance and acceleration for any motion $\mathbf{R}(t)$.

Equation (2) is the computing rule for the velocity $d\mathbf{R}/dt$. It is not the *definition* of $d\mathbf{R}/dt$, which goes back to basics and does not depend on coordinates:

$$\frac{d\mathbf{R}}{dt}=\lim_{\Delta t\to 0}\frac{\Delta\mathbf{R}}{\Delta t}=\lim_{\Delta t\to 0}\frac{\mathbf{R}(t+\Delta t)-\mathbf{R}(t)}{\Delta t}.$$

We repeat: $\mathbf{R}$ is a vector so $\Delta\mathbf{R}$ is a vector so $d\mathbf{R}/dt$ is a vector. All three vectors are in Figure 12.1 ($t$ is not a vector!). This figure reveals the key fact about the geometry: ***The velocity $\mathbf{v}=d\mathbf{R}/dt$ is tangent to the curve.***

The vector $\Delta\mathbf{R}$ goes from one point on the curve to a nearby point. Dividing by $\Delta t$ changes its length, not its direction. That direction lines up with the tangent to the curve, as the points come closer.

**EXAMPLE 1** $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ $\quad \mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$

This curve swings upward as $t$ increases. When $t = 0$ the velocity is $\mathbf{v} = \mathbf{i}$. The tangent is along the $x$ axis, since the $\mathbf{j}$ and $\mathbf{k}$ components are zero. When $t = 1$ the velocity is $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, and the curve is climbing.

For the shadow on the $xy$ plane, drop the $\mathbf{k}$ component. Position on the shadow is $t\mathbf{i} + t^2\mathbf{j}$. Velocity along the shadow is $\mathbf{i} + 2t\mathbf{j}$. The sbadow is a plane curve.

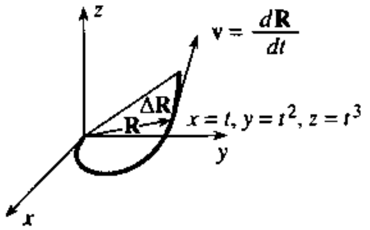



**Fig. 12.1** Position vector $\mathbf{R}$, change $\Delta\mathbf{R}$, velocity $d\mathbf{R}/dt$.

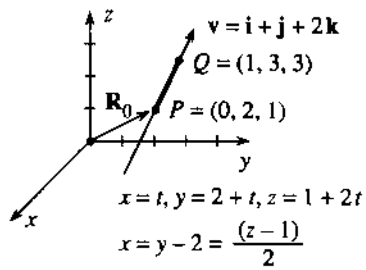



**Fig. 12.2** Equations of a line, with and without the parameter $t$.

**EXAMPLE 2** Uniform motion in a straight line: ***the velocity vector* v *is constant.***

The speed and direction don't change. The position vector moves with $d\mathbf{R}/dt = \mathbf{v}$:

$$\mathbf{R}(t) = \mathbf{R}_0 + t\mathbf{v} \quad (\mathbf{R}_0 \text{ fixed, } \mathbf{v} \text{ fixed, } t \text{ varying}) \tag{3}$$

That is the ***equation of a line*** in vector form. Certainly $d\mathbf{R}/dt = \mathbf{v}$. The starting point $\mathbf{R}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ is given. The velocity $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is also given. Separating the $x$, $y$ and $z$ components, equation (3) for a line is

$$\textit{line with parameter}: \quad x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3. \tag{4}$$

The speed along the line is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. The direction of the line is the unit vector $\mathbf{v}/|\mathbf{v}|$. We have three equations for $x, y, z$, and eliminating $t$ leaves two equations. The parameter $t$ equals $(x - x_0)/v_1$ from equation (4). It also equals $(y - y_0)/v_2$ and $(z - z_0)/v_3$. So these ratios equal each other, and $t$ is gone:

$$\textit{line without parameter}: \quad \frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}. \tag{5}$$

An example is $x = y/2 = z/3$. In this case $(x_0, y_0, z_0) = (0, 0, 0)$—the line goes through the origin. Another point on the line is $(x, y, z) = (2, 4, 6)$. Because $t$ is gone, we cannot say when we reach that point and how fast we are going. The equations $x/4 = y/8 = z/12$ give the same line. Without $t$ we can't know the velocity $\mathbf{v} = d\mathbf{R}/dt$.

**EXAMPLE 3** Find an equation for the line through $P = (0, 2, 1)$ and $Q = (1, 3, 3)$.

**Solution** We have choices! $\mathbf{R}_0$ can go to *any point* on the line. The velocity $\mathbf{v}$ can be *any multiple* of the vector from $P$ to $Q$. The decision on $\mathbf{R}_0$ controls where we start, and $\mathbf{v}$ controls our speed.

The vector from $P$ to $Q$ is $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Those numbers 1, 1, 2 come from subtracting 0, 2, 1 from 1, 3, 3. We choose this vector $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ as a first $\mathbf{v}$, and double it for a second $\mathbf{v}$. We choose the vector $\mathbf{R}_0 = \mathbf{P}$ as a first start and $\mathbf{R}_0 = \mathbf{Q}$ as a second start. Here are two different expressions for the same line—they are $\mathbf{P} + t\mathbf{v}$ and $\mathbf{Q} + t(2\mathbf{v})$:

$$\mathbf{R}(t) = (2\mathbf{j} + \mathbf{k}) + t(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \qquad \mathbf{R}^*(t) = (\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}) + t(2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}).$$

The vector $\mathbf{R}(t)$ gives $x = t$, $y = 2 + t$, $z = 1 + 2t$. The vector $\mathbf{R}^*$ is at a different point on the same line at the same time: $x^* = 1 + 2t$, $y^* = 3 + 2t$, $z^* = 3 + 4t$.

If I pick $t = 1$ in $\mathbf{R}$ and $t = 0$ in $\mathbf{R}^*$, the point is $(1, 3, 3)$. We arrive there at different times. You are seeing how parameters work, to tell "where" and also "when." If $t$ goes from $-\infty$ to $+\infty$, all points on one line are also on the other line. The path is the same, but the "twins" are going at different speeds.

**Question 1** When do these twins meet? When does $\mathbf{R}(t) = \mathbf{R}^*(t)$?
Answer They meet at $t = -1$, when $\mathbf{R} = \mathbf{R}^* = -\mathbf{i} + \mathbf{j} - \mathbf{k}$.

**Question 2** What is an equation for the segment between $P$ and $Q$ (not beyond)?
Answer In the equation for $\mathbf{R}(t)$, let $t$ go from 0 to 1 (not beyond):

$$x = t \quad y = 2 + t \quad z = 1 + 2t \quad [0 \leqslant t \leqslant 1 \text{ for segment}]. \tag{6}$$

At $t = 0$ we start from $P = (0, 2, 1)$. At $t = 1$ we reach $Q = (1, 3, 3)$.

**Question 3** What is an equation for the line without the parameter $t$?
Answer Solve equations (6) for $t$ or use (5): $x/1 = (y - 2)/1 = (z - 1)/2$.

**Question 4** Which point on the line is closest to the origin?
Answer The derivative of $x^2 + y^2 + z^2 = t^2 + (2 + t)^2 + (1 + 2t)^2$ is $8 + 8t$. This derivative is zero at $t = -1$. So the closest point is $(-1, 1, -1)$.

**Question 5** Where does the line meet the plane $x + y + z = 11$?
Answer Equation (6) gives $x + y + z = 3 + 4t = 11$. So $t = 2$. The meeting point is $x = t = 2$, $y = t + 2 = 4$, $z = 1 + 2t = 5$.

**Question 6** What line goes through $(3, 1, 1)$ perpendicular to the plane $x - y - z = 1$?
Answer The normal vector to the plane is $\mathbf{N} = \mathbf{i} - \mathbf{j} - \mathbf{k}$. *That is* $\mathbf{v}$. The position vector to $(3, 1, 1)$ is $\mathbf{R}_0 = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$. Then $\mathbf{R} = \mathbf{R}_0 + t\mathbf{v}$.

## COMPARING LINES AND PLANES

A line has one parameter or two equations. We give the starting point and velocity: $(x, y, z) = (x_0, y_0, z_0) + t(v_1, v_2, v_3)$. That tells directly which points are on the line. Or we eliminate $t$ to find the two equations in (5).

A plane has one equation or two parameters! The equation is $ax + by + cz = d$. That tells us *in*directly which points are on the plane. (Instead of knowing $x, y, z$, we know the equation they satisfy. Instead of directions $\mathbf{v}$ and $\mathbf{w}$ in the plane, we are told the perpendicular direction $\mathbf{N} = (a, b, c)$.) With parameters, the line contains $\mathbf{R}_0 + t\mathbf{v}$ and the plane contains $\mathbf{R}_0 + t\mathbf{v} + s\mathbf{w}$. A plane looks worse with parameters ($t$ and $s$), a line looks better.

Questions 5 and 6 connected lines to planes. Here are two more. See Problems 41–44:

**Question 7** When is the line $\mathbf{R}_0 + t\mathbf{v}$ parallel to the plane? When is it perpendicular?
Answer The test is $\mathbf{v} \cdot \mathbf{N} = 0$. The test is $\mathbf{v} \times \mathbf{N} = 0$.

**EXAMPLE 4** Find the plane containing $P_0 = (1, 2, 1)$ and the line of points $(1, 0, 0) + t(2, 0, -1)$. That vector $\mathbf{v}$ will be in the plane.

Solution The vector $\mathbf{v} = 2\mathbf{i} - \mathbf{k}$ goes along the line. The vector $\mathbf{w} = 2\mathbf{j} + \mathbf{k}$ goes from $(1, 0, 0)$ to $(1, 2, 1)$. Their cross product is

$$\mathbf{N} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}.$$

The plane $2x - 2y + 4z = 2$ has this normal $\mathbf{N}$ and contains the point $(1, 2, 1)$.

## SPEED, DIRECTION, DISTANCE, ACCELERATION

We go back to the curve traced out by $\mathbf{R}(t)$. The derivative $\mathbf{v}(t) = d\mathbf{R}/dt$ is the velocity vector along that curve. The ***speed*** is the magnitude of $\mathbf{v}$:

$$\text{speed} = |\mathbf{v}| = \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}. \tag{7}$$

The ***direction*** of the velocity vector is $\mathbf{v}/|\mathbf{v}|$. This is a unit vector, since $\mathbf{v}$ is divided by its length. ***The unit tangent vector*** $\mathbf{v}/|\mathbf{v}|$ ***is denoted by*** $\mathbf{T}$.

The tangent vector is constant for lines. It changes direction for curves.

**EXAMPLE 5** (important) Find $\mathbf{v}$ and $|\mathbf{v}|$ and $\mathbf{T}$ for steady motion around a circle:

$$x = r \cos \omega t, \quad y = r \sin \omega t, \quad z = 0.$$

Solution The position vector is $\mathbf{R} = r \cos \omega t\, \mathbf{i} + r \sin \omega t\, \mathbf{j}$. The velocity is

$$\mathbf{v} = d\mathbf{R}/dt = -\omega r \sin \omega t\, \mathbf{i} + \omega r \cos \omega t\, \mathbf{j} \quad \text{(tangent, not unit tangent)}$$

The speed is the radius $r$ times the angular velocity $\omega$:

$$|\mathbf{v}| = \sqrt{(-\omega r \sin \omega t)^2 + (\omega r \cos \omega t)^2} = \omega r.$$

The unit tangent vector is $\mathbf{v}$ divided by $|\mathbf{v}|$:

$$\mathbf{T} = -\sin \omega t\, \mathbf{i} + \cos \omega t\, \mathbf{j} \qquad \text{(length 1 since } \sin^2\omega t + \cos^2\omega t = 1).$$

Think next about the *distance traveled*. Distance along a curve is always denoted by $s$ (called ***arc length***). I don't know why we use $s$—certainly not as the initial for speed. In fact speed is distance divided by time. The ratio $s/t$ gives average speed; $ds/dt$ is instantaneous speed. We are back to Chapter 1 and Section 8.3, the relation of speed to distance:

$$\text{speed } |\mathbf{v}| = ds/dt \qquad \text{distance } s = \int (ds/dt)\, dt = \int |\mathbf{v}(t)|\, dt.$$

Notice that $|\mathbf{v}|$ and $s$ and $t$ are scalars. The direction vector is $\mathbf{T}$:

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{R}/dt}{ds/dt} = \frac{d\mathbf{R}}{ds} = \textbf{\textit{unit tangent vector.}} \tag{8}$$

In Figure 12.3, the chord length (straight) is $|\Delta\mathbf{R}|$. The arc length (curved) is $\Delta s$. As $\Delta\mathbf{R}$ and $\Delta s$ approach zero, the ratio $|\Delta\mathbf{R}/\Delta s|$ approaches $|\mathbf{T}| = 1$.

Think finally about the ***acceleration vector*** $\mathbf{a}(t)$. It is the rate of change of velocity (not the rate of change of speed):

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{R}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}. \tag{9}$$

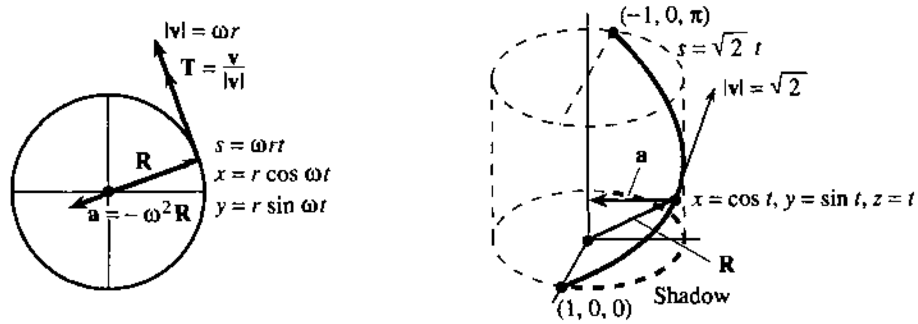



**Fig. 12.3** Steady motion around a circle. Half turn up a helix.

For steady motion along a line, as in $x = t$, $y = 2 + t$, $z = 1 + 2t$, there is no acceleration. The second derivatives are all zero. For steady motion around a circle, there *is* acceleration. In driving a car, you accelerate with the gas pedal or the brake. ***You also accelerate by turning the wheel.*** It is the velocity vector that changes, not the speed.

**EXAMPLE 6** Find the distance $s(t)$ and acceleration $\mathbf{a}(t)$ for circular motion.

Solution The speed in Example 5 is $ds/dt = \omega r$. After integrating, the distance is $s = \omega rt$. At time $t$ we have gone through an angle of $\omega t$. The radius is $r$, so the distance traveled agrees with $\omega t$ times $r$. Note that the dimension of $\omega$ is 1/time. (Angles are dimensionless.) At time $t = 2\pi/\omega$ we have gone once around the circle—to $s = 2\pi r$ not back to $s = 0$.

The acceleration is $\mathbf{a} = d^2\mathbf{R}/dt^2$. Remember $\mathbf{R} = r\cos\omega t\,\mathbf{i} + r\sin\omega t\,\mathbf{j}$:

$$\mathbf{a}(t) = -\omega^2 r\cos\omega t\,\mathbf{i} - \omega^2 r\sin\omega t\,\mathbf{j}. \tag{10}$$

That direction is opposite to $\mathbf{R}$. This is a special motion, with no action on the gas pedal or the brake. All the acceleration is from turning. The magnitude is $|\mathbf{a}| = \omega^2 r$, with the correct dimension of distance/(time)$^2$.

**EXAMPLE 7** Find $\mathbf{v}$ and $s$ and $\mathbf{a}$ around the helix $\mathbf{R} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$.

Solution The velocity is $\mathbf{v} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$. The speed is

$$ds/dt = |\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}\ \text{(constant)}.$$

Then distance is $s = \sqrt{2}\,t$. At time $t = \pi$, a half turn is complete. The distance along the shadow is $\pi$ (a half circle). The distance along the helix is $\sqrt{2}\,\pi$, because of its 45° slope.

The unit tangent vector is velocity/speed, and the acceleration is $d\mathbf{v}/dt$:

$$\mathbf{T} = (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})/\sqrt{2} \qquad \mathbf{a} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}.$$

**EXAMPLE 8** Find $\mathbf{v}$ and $s$ and $\mathbf{a}$ around the ellipse $x = \cos t$, $y = 2\sin t$, $z = 0$.

Solution Take derivatives: $\mathbf{v} = -\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$ and $|\mathbf{v}| = \sqrt{\sin^2 t + 4\cos^2 t}$. This is the speed $ds/dt$. For the distance $s$, something bad happens (or something normal). The speed is not simplified by $\sin^2 t + \cos^2 t = 1$. We cannot integrate $ds/dt$ to find a formula for $s$. The square root defeats us.

The acceleration $-\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$ still points to the center. This is *not* the Earth going around the sun. The path is an ellipse but the speed is wrong. See Section 12.4 (the pound note) for a terrible error in the position of the sun.

**12A** The basic formulas for motion along a curve are

$$\mathbf{v}=\frac{d\mathbf{R}}{dt} \quad \mathbf{a}=\frac{d\mathbf{v}}{dt} \quad |\mathbf{v}|=\frac{ds}{dt} \quad \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{d\mathbf{R}/dt}{ds/dt}=\frac{d\mathbf{R}}{ds}.$$

Suppose we know the acceleration $\mathbf{a}(t)$ and the initial velocity $\mathbf{v}_0$ and position $\mathbf{R}_0$. Then $\mathbf{v}(t)$ and $\mathbf{R}(t)$ are also known. We integrate each component:

$$\mathbf{a}(t)=\text{constant} \Rightarrow \mathbf{v}(t)=\mathbf{v}_0+\mathbf{a}t \qquad \Rightarrow \mathbf{R}(t)=\mathbf{R}_0+\mathbf{v}_0 t+\tfrac{1}{2}\mathbf{a}t^2$$

$$\mathbf{a}(t)=\cos t\ \mathbf{k} \quad \Rightarrow \mathbf{v}(t)=\mathbf{v}_0+\sin t\ \mathbf{k} \Rightarrow \mathbf{R}(t)=\mathbf{R}_0+\mathbf{v}_0 t-\cos t\ \mathbf{k}.$$

### THE CURVE OF A BASEBALL

There is a nice discussion of curve balls in the calculus book by Edwards and Penney. We summarize it here (optionally). The ball leaves the pitcher's hand five feet off the ground: $\mathbf{R}_0=0\mathbf{i}+0\mathbf{j}+5\mathbf{k}$. The initial velocity is $\mathbf{v}_0=120\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ (120 ft/sec is more than 80 miles per hour). The acceleration is $-32\mathbf{k}$ from gravity, plus a new term from ***spin***. If the spin is around the $z$ axis, and the ball goes along the $x$ axis, then this acceleration is in the $y$ direction. (It comes from the cross product $\mathbf{k}\times\mathbf{i}$—there is a pressure difference on the sides of the ball.) A good pitcher can achieve $\mathbf{a}=16\mathbf{j}-32\mathbf{k}$. The batter integrates as fast as he can:

$$\mathbf{v}(t)=\mathbf{v}_0+\mathbf{a}t=120\mathbf{i}+(-2+16t)\mathbf{j}+(2-32t)\mathbf{k}$$

$$\mathbf{R}(t)=\mathbf{R}_0+\mathbf{v}_0 t+\tfrac{1}{2}\mathbf{a}t^2=120t\ \mathbf{i}+(-2t+8t^2)\mathbf{j}+(5+2t-16t^2)\mathbf{k}.$$

Notice the $t^2$. The effect of spin is small at first, then suddenly bigger (as every batter knows). So is the effect of gravity—the ball starts to dive. At $t=\frac{1}{2}$, the $\mathbf{i}$ component is 60 feet and the ball reaches the batter. The $\mathbf{j}$ component is 1 foot and the $\mathbf{k}$ component is 2 feet—the curve goes low over the outside corner.

At $t=\frac{1}{4}$, when the batter saw the ball halfway, the $\mathbf{j}$ component was zero. It looked as if it was coming right over the plate.

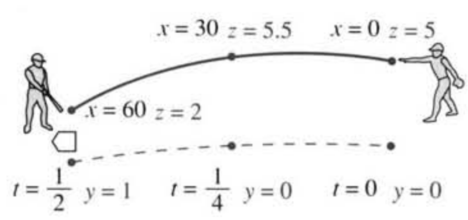



Fig. 12.4 A curve ball approaches home plate. Halfway it is on line.

## 12.1 EXERCISES

**Read-through questions**

The position vector __a__ along the curve changes with the parameter $t$. The velocity is __b__. The acceleration is __c__. If the position is $\mathbf{i}+t\mathbf{j}+t^2\mathbf{k}$, then $\mathbf{v}=$ __d__ and $\mathbf{a}=$ __e__. In that example the speed is $|\mathbf{v}|=$ __f__. This equals $ds/dt$, where $s$ measures the __g__. Then $s=\int$ __h__. The tangent vector is in the same direction as the __i__, but $\mathbf{T}$ is a __j__ vector. In general $\mathbf{T}=$ __k__ and in the example $\mathbf{T}=$ __l__.

Steady motion along a line has $\mathbf{a}=$ __m__. If the line is $x=y=z$, the unit tangent vector is $\mathbf{T}=$ __n__. If the speed is

$|\mathbf{v}|=\sqrt{3}$, the velocity vector is $\mathbf{v}=$ __o__. If the initial position is (1, 0, 0), the position vector is $\mathbf{R}(t)=$ __p__. The general equation of a line is $x=x_0+tv_1$, $y=$ __q__, $z=$ __r__. In vector notation this is $\mathbf{R}(t)=$ __s__. Eliminating $t$ leaves the equations $(x-x_0)/v_1=(y-y_0)/v_2=$ __t__. A line in space needs __u__ equations where a plane needs __v__. A line has one parameter where a plane has __w__. The line from $\mathbf{R}_0=(1,0,0)$ to (2, 2, 2) with $|\mathbf{v}|=3$ is $\mathbf{R}(t)=$ __x__.

Steady motion around a circle (radius $r$, angular velocity $\omega$) has $x=$ __y__, $y=$ __z__, $z=0$. The velocity is $\mathbf{v}=$ __A__. The speed is $|\mathbf{v}|=$ __B__. The acceleration is $\mathbf{a}=$ __C__, which has magnitude __D__ and direction __E__. Combining upward motion $\mathbf{R}=t\mathbf{k}$ with this circular motion produces motion around a __F__. Then $\mathbf{v}=$ __G__ and $|\mathbf{v}|=$ __H__.

**1** Sketch the curve with parametric equations $x=t$, $y=t^3$. Find the velocity vector and the speed at $t=1$.

**2** Sketch the path with parametric equations $x=1+t$, $y=1-t$. Find the $xy$ equation of the path and the speed along it.

**3** On the circle $x=\cos t$, $y=\sin t$ explain by the chain rule and then by geometry why $dy/dx=-\cot t$.

**4** Locate the highest point on the curve $x=6t$, $y=6t-t^2$. This curve is a ________. What is the acceleration $\mathbf{a}$?

**5** Find the velocity vector and the $xy$ equation of the tangent line to $x=e^t$, $y=e^{-t}$ at $t=0$. What is the $xy$ equation of the curve?

**6** Describe the shapes of these curves: (a) $x=2^t$, $y=4^t$; (b) $x=4^t$, $y=8^t$; (c) $x=4^t$, $y=4t$.

**Note:** *To find* "parametric equations" *is to find* $x(t)$, $y(t)$, *and possibly* $z(t)$.

**7** Find parametric equations for the line through $P=(1,2,4)$ and $Q=(5,5,4)$. Probably your speed is 5; change the equations so the speed is 10. Probably your $\mathbf{R}_0$ is $P$; change the start to $Q$.

**8** Find an equation for any one plane that is perpendicular to the line in Problem 7. Also find equations for any one line that is perpendicular.

**9** On a straight line from (2, 3, 4) with velocity $\mathbf{v}=\mathbf{i}-\mathbf{k}$, the position vector is $\mathbf{R}(t)=$ ________. If the velocity vector is changed to $t\mathbf{i}-t\mathbf{k}$, then $\mathbf{R}(t)=$ ________. The path is still ________.

**10** Find parametric equations for steady motion from $P=(3,1,-2)$ at $t=0$ on a line to $Q=(0,0,0)$ at $t=3$. What is the speed? Change parameters so the speed is $e^t$.

**11** The equations $x-1=\frac{1}{2}(y-2)=\frac{1}{3}(z-2)$ describe a ________. The same path is given parametrically by $x=1+t$, $y=$ ________, $z=$ ________. The same path is also given by $x=1+2t$, $y=$ ________, $z=$ ________.

**12** Find parametric equations to go around the unit circle with speed $e^t$ starting from $x=1$, $y=0$. When is the circle completed?

**13** The path $x=2y=3z=6t$ is a ________ traveled with speed ________. If $t$ is restricted by $t\geqslant 1$ the path starts at ________. If $t$ is restricted by $0\leqslant t\leqslant 1$ the path is a ________.

**14** Find the closest point to the origin on the line $x=1+t$, $y=2-t$. When and where does it cross the 45° line through the origin? Find the equation of a line it never crosses.

**15** (a) How far apart are the two parallel lines $x=y$ and $x=y+1$? (b) How far is the point $x=t$, $y=t$ from the point $x=t$, $y=t+1$? (c) What is the closest distance if their speeds are different: $x=t$, $y=t$ and $x=2t$, $y=2t+1$?

**16** Which vectors follow the same path as $\mathbf{R}=t\mathbf{i}+t^2\mathbf{j}$? The speed along the path may be different.

(a) $2t\mathbf{i}+2t^2\mathbf{j}$ (b) $2t\mathbf{i}+4t^2\mathbf{j}$ (c) $-t\mathbf{i}+t^2\mathbf{j}$ (d) $t^3\mathbf{i}+t^6\mathbf{j}$

**17** Find a parametric form for the straight line $y=mx+b$.

**18** The line $x=1+v_1t$, $y=2+v_2t$ passes through the origin provided ________ $v_1+$ ________ $v_2=0$. This line crosses the 45° line $y=x$ unless ________ $v_1+$ ________ $v_2=0$.

**19** Find the velocity $\mathbf{v}$ and speed $|\mathbf{v}|$ and tangent vector $\mathbf{T}$ for these motions: (a) $\mathbf{R}=t\mathbf{i}+t^{-1}\mathbf{j}$ (b) $\mathbf{R}=t\cos t\,\mathbf{i}+t\sin t\,\mathbf{j}$ (c) $\mathbf{R}=(t+1)\mathbf{i}+(2t+1)\mathbf{j}+(2t+2)\mathbf{k}$.

**20** If the velocity $dx/dt\,\mathbf{i}+dy/dt\,\mathbf{j}$ is always perpendicular to the position vector $x\mathbf{i}+y\mathbf{j}$, show from their dot product that $x^2+y^2$ is constant. The point stays on a circle.

**21** Find two paths $\mathbf{R}(t)$ with the same $\mathbf{v}=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}$. Find a third path with a different $\mathbf{v}$ but the same acceleration.

**22** If the acceleration is a constant vector, the path must be ________. If the path is a straight line, the acceleration vector must be ________.

**23** Find the minimum and maximum speed if $x=t+\cos t$, $y=t-\sin t$. Show that $|\mathbf{a}|$ is constant but not $\mathbf{a}$. The point is going around a circle while the center is moving on what line?

**24** Find $x(t)$, $y(t)$ so that the point goes around the circle $(x-1)^2+(y-3)^2=4$ with speed 1.

**25** A ball that is circling with $x=\cos 2t$, $y=\sin 2t$ flies off on a tangent at $t=\pi/8$. Find its departure point and its position at a later time $t$ (linear motion; compute its constant velocity $\mathbf{v}$).

**26** Why is $|\mathbf{a}|$ generally different from $d^2s/dt^2$? Give an example of the difference, and an example where they are equal.

**27** Change $t$ so that the speed along the helix $\mathbf{R}=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+t\,\mathbf{k}$ is 1 instead of $\sqrt{2}$. Call the new parameter $s$.

**28** Find the speed $ds/dt$ on the line $x=1+6t$, $y=2+3t$, $z=2t$. Integrate to find the length $s$ from (1, 2, 0) to (13, 8, 4). Check by using $12^2+6^2+4^2$.

**29** Find $\mathbf{v}$ and $|\mathbf{v}|$ and $\mathbf{a}$ for the curve $x = \tan t$, $y = \sec t$. What is this curve? At what time does it go to infinity, and along what line?

**30** Construct parametric equations for travel on a helix with speed $t$.

**31** Suppose the unit tangent vector $\mathbf{T}(t)$ is the derivative of $\mathbf{R}(t)$. What does that say about the speed? Give a noncircular example.

**32** For travel on the path $y = f(x)$, with no parameter, it is impossible to find the ________ but still possible to find the ________ at each point of the path.

**Find $x(t)$ and $y(t)$ for paths 33–36.**

**33** Around the square bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, with speed 2. The formulas have four parts.

**34** Around the unit circle with speed $e^{-t}$. Do you get all the way around?

**35** Around a circle of radius 4 with acceleration $|\mathbf{a}| = 1$.

**36** Up and down the $y$ axis with constant acceleration $-\mathbf{j}$, returning to $(0, 0)$ at $t = 10$.

**37** True (with reason) or false (with example):

(a) If $|\mathbf{R}| = 1$ for all $t$ then $|\mathbf{v}| = \text{constant}$.

(b) If $\mathbf{a} = 0$ then $\mathbf{R} = \text{constant}$.

(c) If $\mathbf{v} \cdot \mathbf{v} = \text{constant}$ then $\mathbf{v} \cdot \mathbf{a} = 0$.

(d) If $\mathbf{v} \cdot \mathbf{R} = 0$ then $\mathbf{R} \cdot \mathbf{R} = \text{constant}$.

(e) There is no path with $\mathbf{v} = \mathbf{a}$.

**38** Find the position vector to the shadow of $t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ on the $xz$ plane. Is the curve ever parallel to the line $x = y = z$?

**39** On the ellipse $x = a\cos t$, $y = b\sin t$, the angle $\theta$ from the center is not the same as $t$ because ________.

**40** Two particles are racing from $(1, 0)$ to $(0, 1)$. One follows $x = \cos t$, $y = \sin t$, the other follows $x = 1 + v_1 t$, $y = v_2 t$. Choose $v_1$ and $v_2$ so that the second particle goes slower but wins.

**41** Two lines in space are given by $\mathbf{R}(t) = \mathbf{P} + t\mathbf{v}$ and $\mathbf{R}(t) = \mathbf{Q} + t\mathbf{w}$. Four possibilities: The lines are parallel or the same or intersecting or skew. Decide which is which based on the vectors $\mathbf{v}$ and $\mathbf{w}$ and $\mathbf{u} = \mathbf{Q} - \mathbf{P}$ (which goes between the lines):

(a) The lines are parallel if ________ are parallel.

(b) The lines are the same if ________ are parallel.

(c) The lines intersect if ________ are not parallel but ________ lie in the same plane.

(d) The lines are skew if the triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is ________.

**42** If the lines are skew (not in the same plane), find a formula based on $\mathbf{u}, \mathbf{v}, \mathbf{w}$ for the distance between them. The vector $\mathbf{u}$ may not be perpendicular to the two lines, so project it onto a vector that is.

**43** The distance from $\mathbf{Q}$ to the line $\mathbf{P} + t\mathbf{v}$ is the projection of $\mathbf{u} = \mathbf{Q} - \mathbf{P}$ perpendicular to $\mathbf{v}$. How far is $\mathbf{Q} = (9, 4, 5)$ from the line $x = 1 + t$, $y = 1 + 2t$, $z = 3 + 2t$?

**44** Solve Problem 43 by calculus: substitute for $x, y, z$ in $(x - 9)^2 + (y - 4)^2 + (z - 5)^2$ and minimize. Which $(x, y, z)$ on the line is closest to $(9, 4, 5)$?

**45** Practice with parameters, starting from $x = F(t)$, $y = G(t)$.

(a) The mirror image across the 45° line is $x =$ ________, $y =$ ________.

(b) Write the curve $x = t^3$, $y = t^2$ as $y = f(x)$.

(c) Why can't $x = t^2$, $y = t^3$ be written as $y = f(x)$?

(d) If $F$ is invertible then $t = F^{-1}(x)$ and $y =$ ________$(x)$.

**46** From 12:00 to 1:00 a snail crawls steadily out the minute hand (one meter in one hour). Find its position at time $t$ starting from $(0, 0)$.

## 12.2 Plane Motion: Projectiles and Cycloids

The previous section started with $\mathbf{R}(t)$. From this position vector we computed $\mathbf{v}$ and $\mathbf{a}$. Now we find $\mathbf{R}(t)$ itself, from more basic information. The laws of physics govern projectiles, and the motion of a wheel produces a cycloid (which enters problems in robotics). The projectiles fly without friction, so the only force is gravity.

These motions occur in a plane. The two components of position will be $x$ (across) and $y$ (up). A projectile moves as $t$ changes, so we look for $x(t)$ and $y(t)$. We are shooting a basketball or firing a gun or peacefully watering the lawn, and we have to aim in the right direction (not directly at the target). If the hose delivers water at 10 meters/second, can you reach the car 12 meters away?

The usual initial position is (0, 0). Some flights start higher, at $(0, h)$. The initial velocity is $(v_0 \cos\alpha,\ v_0 \sin\alpha)$, where $v_0$ is the speed and $\alpha$ is the angle with the horizontal. The acceleration from gravity is purely vertical: $d^2y/dt^2 = -g$. So the horizontal velocity stays at its initial value. The upward velocity decreases by $-gt$:

$$dx/dt = v_0 \cos\alpha,\ dy/dt = v_0 \sin\alpha - gt.$$

The horizontal distance $x(t)$ is steadily increasing. The height $y(t)$ increases and then decreases. To find the position, integrate the velocities (for a high start add $h$ to $y$):

$$\textbf{\textit{The projectile path is}}\ x(t) = (v_0 \cos\alpha)t,\ y(t) = (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2. \qquad (1)$$

This path is a *parabola*. But it is not written as $y = ax^2 + bx + c$. It could be, if we eliminated $t$. Then we would lose track of time. The parabola is $y(x)$, with no parameter, where we have $x(t)$ and $y(t)$.

Basic question: ***Where does the projectile hit the ground***? For the parabola, we solve $y(x) = 0$. That gives the position $x$. *For the projectile we solve* $y(t) = 0$. That gives the *time* it hits the ground, not the place. If that time is $T$, then $x(T)$ gives the place. The information is there. It takes two steps instead of one, but we learn more.

**EXAMPLE 1** Water leaves the hose at 10 meters/second (this is $v_0$). It starts up at the angle $\alpha$. Find the time $T$ when $y$ is zero again, and find where the projectile lands.

Solution The flight ends when $y = (10 \sin\alpha)T - \frac{1}{2}gT^2 = 0$. The flight time is $T = (20 \sin\alpha)/g$. At that time, the horizontal distance is

$$x(T) = (10 \cos\alpha)T = (200 \cos\alpha \sin\alpha)/g. \text{ This is the } \textbf{\textit{range}}\ R.$$

The projectile (or water from the hose) hits the ground at $x = R$. To simplify, replace $200 \cos\alpha \sin\alpha$ by $100 \sin 2\alpha$. Since $g = 9.8$ meters/sec$^2$, ***we can't reach the car***:

$$\text{The range } R = (100 \sin 2\alpha)/9.8 \text{ is at most } 100/9.8. \text{ This is less than 12.}$$

The range is greatest when $\sin 2\alpha = 1$ ($\alpha$ is 45°). To reach 12 meters we could stand on a ladder (Problem 14). To hit a baseball against air resistance, the best angle is nearer to 35°. Figure 12.5 shows symmetric parabolas (no air resistance) and unsymmetric flight paths that drop more steeply.

> **12B** The flight time $T$ and the horizontal range $R = x(T)$ are reached when $y = 0$, which means $(v_0 \sin\alpha)T = \frac{1}{2}gT^2$:
>
> $$T = (2v_0 \sin\alpha)/g \text{ and } R = (v_0 \cos\alpha)T = (v_0^2 \sin 2\alpha)/g.$$

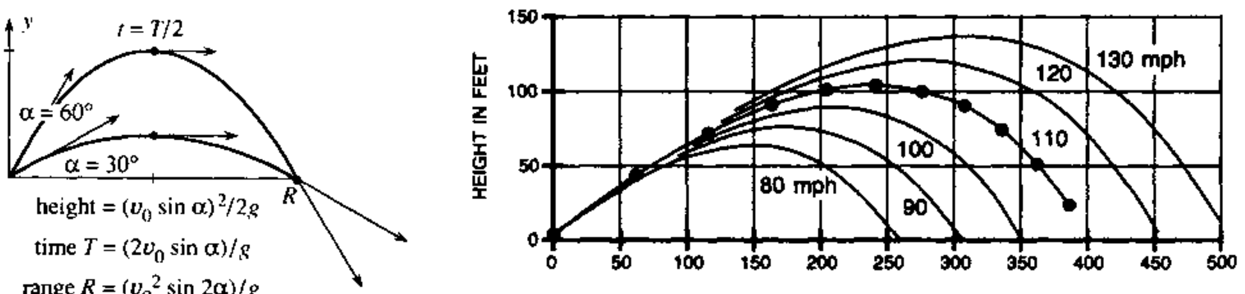



**Fig. 12.5** Equal range $R$, different times $T$. Baseballs hit at 35° with increasing $v_0$. The dots are at half-seconds (from *The Physics of Baseball* by Robert Adair: Harper and Row 1990).

**EXAMPLE 2** What are the correct angles $\alpha$ for a given range $R$ and given $v_0$?

Solution The range is $R=(v_0^2 \sin 2\alpha)/g$. This determines the sine of $2\alpha$—***but two angles can have the same sine.*** We might find $2\alpha = 60°$ or $120°$. The starting angles $\alpha = 30°$ and $\alpha = 60°$ in Figure 12.5 give the same $\sin 2\alpha$ and the same range $R$. The flight times contain $\sin\alpha$ and are different.

By calculus, the maximum height occurs when $dy/dt = 0$. Then $v_0 \sin\alpha = gt$, which means that $t = (v_0 \sin\alpha)/g$. This is half of the total flight time $T$—the time going up equals the time coming down. The value of $y$ at this halfway time $t = \frac{1}{2}T$ is

$$y_{\max} = (v_0 \sin\alpha)(v_0 \sin\alpha)/g - \tfrac{1}{2}g(v_0 \sin\alpha/g)^2 = (v_0 \sin\alpha)^2/2g. \tag{2}$$

**EXAMPLE 3** If a ski jumper goes 90 meters down a 30° slope, after taking off at 28 meters/second, find equations for the flight time and the ramp angle $\alpha$.

Solution The jumper lands at the point $x = 90\cos 30°$, $y = -90 \sin 30°$ (minus sign for obvious reasons). The basic equation (2) is $x = (28\cos\alpha)t$, $y = (28 \sin\alpha)t - \frac{1}{2}gt^2$. Those are two equations for $\alpha$ and $t$. Note that $t$ is not $T$, the flight time to $y = 0$.

***Conclusion*** The position of a projectile involves three parameters $v_0$, $\alpha$, and $t$. ***Three pieces of information determine the flight*** (almost). The reason for the word *almost* is the presence of $\sin\alpha$ and $\cos\alpha$. Some flight requirements cannot be met (reaching a car at 12 meters). Other requirements can be met in two ways (when the car is close). The equation $\sin\alpha = c$ is more likely to have no solution or two solutions than exactly one solution.

Watch for the three pieces of information in each problem. When a football starts at $v_0 = 20$ meters/second and hits the ground at $x = 40$ meters, the third fact is ______. This is like a lawyer who is asked the fee and says \$1000 for three questions. "Isn't that steep?" says the client. "Yes," says the lawyer, "now what's your last question?"

## CYCLOIDS

A projectile's path is a parabola. To compute it, eliminate $t$ from the equations for $x$ and $y$. Problem 5 finds $y = ax^2 + bx$, a parabola through the origin. The path of a point on a wheel seems equally simple, but eliminating $t$ is virtually impossible. The cycloid is a curve that really needs and uses a parameter.

To trace out a cycloid, ***roll a circle of radius a along the x axis.*** Watch the point that starts at the bottom of the circle. It comes back to the bottom at $x = 2\pi a$, after a complete turn of the circle. The path in between is shown in Figure 12.6. After a century of looking for the $xy$ equation, a series of great scientists (Galileo, Christopher Wren, Huygens, Bernoulli, even Newton and l'Hôpital) found the right way to study a cycloid—by introducing a parameter. We will call it $\theta$; it could also be $t$.

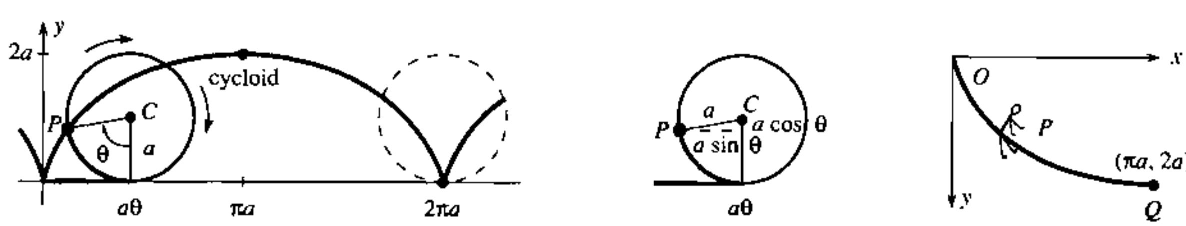



**Fig. 12.6** Path of $P$ on a rolling circle is a cycloid. Fastest slide to Q.

The parameter is the angle $\theta$ through which the circle turns. (This angle is not at the origin, like $\theta$ in polar coordinates.) The circle rolls a distance $a\theta$, radius times angle, along the $x$ axis. So the center of the circle is at $x = a\theta$, $y = a$. To account for the segment $CP$, subtract $a \sin\theta$ from x and $a\cos\theta$ from $y$:

$$\textbf{\textit{The point P has }} x = a(\theta - \sin\theta) \textbf{\textit{ and }} y = a(1 - \cos\theta). \tag{3}$$

At $\theta = 0$ the position is $(0, 0)$. At $\theta = 2\pi$ the position is $(2\pi a, 0)$. In between, the slope of the cycloid comes from the chain rule:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a \sin\theta}{a(1 - \cos\theta)}. \tag{4}$$

This is infinite at $\theta = 0$. The point on the circle starts straight upward and the cycloid has a *cusp*. Note how all calculations use the parameter $\theta$. We go quickly:

**Question 1** Find the area under one arch of the cycloid ($\theta = 0$ to $\theta = 2\pi$).
Answer The area is $\int y\,dx = \int_0^{2\pi} a(1 - \cos\theta)a(1 - \cos\theta)d\theta$. This equals $3\pi a^2$.

**Question 2** Find the length of the arch, using $ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta$.
Answer $\int ds = \int_0^{2\pi} a\sqrt{(1 - \cos\theta)^2 + (\sin\theta)^2}\,d\theta = \int_0^{2\pi} a\sqrt{2 - 2\cos\theta}\,d\theta$.
Now substitute $1 - \cos\theta = 2\sin^2\frac{1}{2}\theta$. The square root is $2 \sin\frac{1}{2}\theta$. The length is $8a$.

**Question 3** If the cycloid is turned over ($y$ is downward), find the time to slide to the bottom. The slider starts with $v = 0$ at $y = 0$.
Answer Kinetic plus potential energy is $\frac{1}{2}mv^2 - mgy = 0$ (it starts from zero and can't change). So the speed is $v = \sqrt{2gy}$. This is $ds/dt$ and we know $\boldsymbol{ds}$:

$$\text{sliding time} = \int dt = \int \frac{ds}{\sqrt{2gy}} = \int_0^{\pi} \frac{a\sqrt{2 - 2\cos\theta}\,d\theta}{\sqrt{2ga(1 - \cos\theta)}} = \pi\sqrt{a/g}.$$

Check dimensions: $a$ = distance, $g$ = distance/(time)$^2$, $\pi\sqrt{a/g}$ = time. ***That is the shortest sliding time for any curve.*** The cycloid solves the "brachistochrone problem," which minimizes the time down curves from $O$ to $Q$ (Figure 12.6). You might think a straight path would be quicker—it is certainly shorter. A straight line has the equation $x = \pi y/2$, so the sliding time is

$$\int dt = \int ds/\sqrt{2gy} = \int_0^{2a} \sqrt{(\pi/2)^2 + 1}\,dy/\sqrt{2gy} = \sqrt{\pi^2 + 4}\sqrt{a/g}. \tag{5}$$

This is larger than the cycloid time $\pi\sqrt{a/g}$. It is better to start out vertically and pick up speed early, even if the path is longer.

Instead of publishing his solution, John Bernoulli turned this problem into an international challenge: *Prove that the cycloid gives the fastest slide.* Most mathematicians couldn't do it. The problem reached Isaac Newton (this was later in his life). As you would expect, Newton solved it. For some reason he sent back his proof with no name. But when Bernoulli received the answer, he was not fooled for a moment: "I recognize the lion by his claws."

What is also amazing is a further property of the cycloid: ***The time to Q is the same if you begin anywhere along the path.*** Starting from rest at $P$ instead of $O$, the bottom is reached at the same time. This time Bernoulli got carried away: "You will be petrified with astonishment when I say...".

There are other beautiful curves, closely related to the cycloid. For an ***epicycloid***, the circle rolls around the outside of another circle. For a ***hypocycloid***, the rolling circle is inside the fixed circle. The *astroid* is the special case with radii in the ratio 1 to 4. It is the curved star in Problem 34, where $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

The cycloid even solves the old puzzle: *What point moves backward when a train starts forward*? The train wheels have a flange that extends below the track, and $dx/dt < 0$ at the bottom of the flange.

## 12.2 EXERCISES

**Read-through questions**

A projectile starts with speed $v_0$ and angle $\alpha$. At time $t$ its velocity is $dx/dt =$ __a__, $dy/dt =$ __b__ (the downward acceleration is $g$). Starting from $(0, 0)$, the position at time $t$ is $x =$ __c__, $y =$ __d__. The flight time back to $y = 0$ is $T =$ __e__. At that time the horizontal range is $R =$ __f__. The flight path is a __g__.

The three quantities $v_0$, __h__, __i__ determine the projectile's motion. Knowing $v_0$ and the position of the target, we (can) (cannot) solve for $\alpha$. Knowing $\alpha$ and the position of the target, we (can) (cannot) solve for $v_0$.

A __j__ is traced out by a point on a rolling circle. If the radius is $a$ and the turning angle is $\theta$, the center of the circle is at $x =$ __k__, $y =$ __l__. The point is at $x =$ __m__, $y =$ __n__, starting from $(0, 0)$. It travels a distance __o__ in a full turn of the circle. The curve has a __p__ at the end of every turn. An upside-down cycloid gives the __q__ slide between two points.

**Problems 1–18 and 41 are about projectiles**

**1** Find the time of flight $T$, the range $R$, and the maximum height $Y$ of a projectile with $v_0 = 16$ ft/sec and

(a) $\alpha = 30°$  (b) $\alpha = 60°$  (c) $\alpha = 90°$.

**2** If $v_0 = 32$ ft/sec and the projectile returns to the ground at $T = 1$, find the angle $\alpha$ and the range $R$.

**3** A ball is thrown at 60° with $v_0 = 20$ meters/sec to clear a wall 2 meters high. How far away is the wall?

**4** If $\mathbf{v}(0) = 3\mathbf{i} + 3\mathbf{j}$ find $\mathbf{v}(t)$, $\mathbf{v}(1)$, $\mathbf{v}(2)$ and $\mathbf{R}(t)$, $\mathbf{R}(1)$, $\mathbf{R}(2)$.

**5** (a) Eliminate $t$ from $x = t$, $y = t - \frac{1}{2}t^2$ to find the $xy$ equation of the path. At what $x$ is $y = 0$?
(b) Do the same for any $v_0$ and $\alpha$.

**6** Find the angle $\alpha$ for a ball kicked at 30 meters/second if it clears 6 meters traveling horizontally.

**7** How far out does a stone hit the water $h$ feet below, starting with velocity $v_0$ at angle $\alpha = 0$?

**8** How far out does the same stone go, starting at angle $\alpha$? Find an equation for the angle that maximizes the range.

**9** A ball starting from $(0, 0)$ passes through $(5, 2)$ after 2 seconds. Find $v_0$ and $\alpha$. (The units are meters.)

***10** With $x$ and $y$ from equation (1), show that

$$v_0^2 \geq (gx/v_0)^2 + 2gy.$$

If a fire is at height $H$ and the water velocity is $v_0$, how far can the fireman put the hose back from the fire? (The parabola in this problem is the "envelope" enclosing all possible paths.)

**11** Estimate the initial speed of a 100-meter golf shot hit at $\alpha = 45°$. Is the true $v_0$ larger or smaller, when air friction is included?

**12** $T = 2v_0(\sin\alpha)/g$ is in seconds and $R = (v_0^2 \sin 2\alpha)/g$ is in meters if $v_0$ and $g$ are in ________.

**13** (a) What is the greatest height a ball can be thrown? Aim straight up with $v_0 = 28$ meters/sec.

**14** If a baseball goes 100 miles per hour for 60 feet, how long does it take (in seconds) and how far does it fall from gravity (in feet)? Use $\frac{1}{2}gt^2$.

**15** If you double $v_0$, what happens to the range and maximum height? If you change the angle by $d\alpha$, what happens to those numbers?

**16** At what point on the path is the speed of the projectile (a) least (b) greatest?

**17** If the hose with $v_0 = 10$m/sec is at a 45° angle, $x$ reaches 12 meters when $t =$ ________ and $y =$ ________. From a ladder of height ________ the water will reach the car (12 meters).

**18** Describe the two trajectories a golf ball can take to land right in the hole, if it starts with a large known velocity $v_0$. In reality (with air resistance) which of those shots would fall closer?

**Problems 19–34 are about cycloids and related curves**

**19** Find the unit tangent vector $\mathbf{T}$ to the cycloid. Also find the speed at $\theta = 0$ and $\theta = \pi$, if the wheel turns at $d\theta/dt = 1$.

**20** The slope of the cycloid is infinite at $\theta = 0$:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{1 - \cos\theta}.$$

By whose rule? Estimate the slope at $\theta = \frac{1}{10}$ and $\theta = -\frac{1}{10}$. Where does the slope equal one?

**21** Show that the tangent to the cycloid at $P$ in Figure 12.6a goes through $x = a\theta$, $y = 2a$. Where is this point on the rolling circle?

**22** For a **trochoid**, the point $P$ is a distance $d$ from the center of the rolling circle. Redraw Figure 12.6b to find $x = a\theta - d\sin\theta$ and $y =$ ________.

**23** If a circle of radius $a$ rolls inside a circle of radius $2a$, show that one point on the small circle goes across on a straight line.

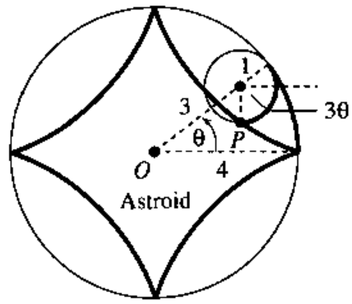
**24** Find $d^2y/dx^2$ for the cycloid, which is concave ________.

**25** If $d\theta/dt = c$, find the velocities $dx/dt$ and $dy/dt$ along the cycloid. Where is $dx/dt$ greatest and where is $dy/dt$ greatest?

**26** Experiment with graphs of $x = a\cos\theta + b\sin\theta$, $y = c\cos\theta + d\sin\theta$ using a computer. What kind of curves are they? Why are they closed?

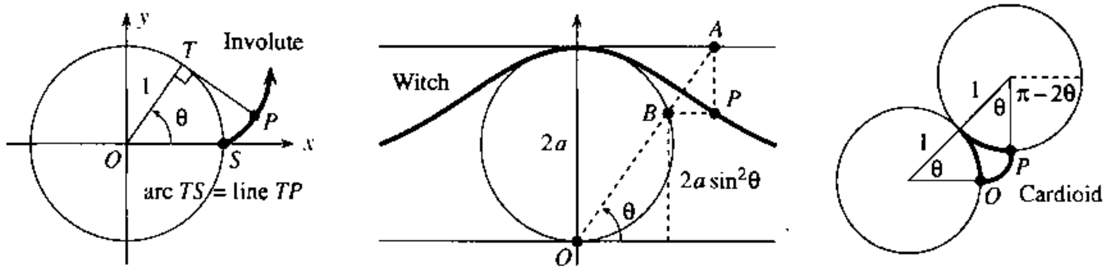
**27** A stone in a bicycle tire goes along a cycloid. Find equations for the stone's path if it flies off at the top (a projectile).

**28** Draw curves on a computer with $x = a\cos\theta + b\cos 3\theta$ and $y = c\sin\theta + d\sin 3\theta$. Is there a limit to the number of loops?




**35** Find the area inside the astroid.

**36** Explain why $x = 2a\cot\theta$ and $y = 2a\sin^2\theta$ for the point $P$ on the **witch of Agnesi**. Eliminate $\theta$ to find the $xy$ equation. **Note**: Maria Agnesi wrote the first three-semester calculus text (l'Hôpital didn't do integral calculus). The word "witch" is a total mistranslation, nothing to do with her or the curve.




**29** When a penny rolls completely around another penny, the head makes ________ turns. When it rolls inside a circle four times larger (for the astroid), the head makes ________ turns.

**30** Display the cycloid family with computer graphics:

(a) **cycloid**

(b) **epicycloid** $x = C\cos\theta - \cos C\theta$, $y = C\sin\theta + \sin C\theta$

(c) **hypocycloid** $x = c\cos\theta + \cos c\theta$, $y = c\sin\theta - \sin c\theta$

(d) **astroid** ($c = 3$)

(e) **deltoid** ($c = 2$).

**31** If one arch of the cycloid is revolved around the $x$ axis, find the surface area and volume.

**32** For a hypocycloid the fixed circle has radius $c + 1$ and the circle rolling inside has radius 1. There are $c + 1$ cusps if $c$ is an integer. How many cusps (use computer graphics if possible) for $c = 1/2$? $c = 3/2$? $c = \sqrt{2}$? *What curve for* $c = 1$?

**33** When a string is unwound from a circle find $x(\theta)$ and $y(\theta)$ for point $P$. Its path is the "**involute**" of the circle.

**34** For the point $P$ on the **astroid**, explain why $x = 3\cos\theta + \cos 3\theta$ and $y = 3\sin\theta - \sin 3\theta$. The angle in the figure is $3\theta$ because both circular arcs have length ________. Convert to $x = 4\cos^3\theta$, $y = 4\sin^3\theta$ by triple-angle formulas.

**37** For a **cardioid** the radius $C - 1$ of the fixed circle equals the radius 1 of the circle rolling outside (epicycloid with $C = 2$). (a) The coordinates of $P$ are $x = -1 + 2\cos\theta - \cos 2\theta$, $y =$ ________. (b) The double-angle formulas yield $x = 2\cos\theta(1 - \cos\theta)$, $y =$ ________. (c) $x^2 + y^2 =$ ________ so its square root is $r =$ ________.

**38** Explain the last two steps in equation (5) for the sliding time down a straight path.

**39** On an upside-down cycloid the slider takes the same time $T$ to reach bottom *wherever it starts*. Starting at $\theta = \alpha$, write $1 - \cos\theta = 2\sin^2\theta/2$ and $1 - \cos\alpha = 2\sin^2\alpha/2$ to show that

$$T = \int_\alpha^\pi \frac{\sqrt{2a^2(1-\cos\theta)}\,d\theta}{\sqrt{2ag(\cos\alpha - \cos\theta)}} = \pi\sqrt{\frac{a}{g}}.$$

**40** Suppose a heavy weight is attached to the top of the rolling circle. What is the path of the weight?

**41** The wall in Fenway Park is 37 feet high and 315 feet from home plate. A baseball hit 3 feet above the ground at $\alpha = 22.5°$ will just go over if $v_0 =$ ________. The time to reach the wall is ________.

## 12.3 Curvature and Normal Vector

A driver produces acceleration three ways—by the gas pedal, the brake, and steering wheel. The first two change the speed. Turning the wheel changes the direction. *All three change the velocity* (they give acceleration). For steady motion around a circle, the change is from steering—the acceleration $d\mathbf{v}/dt$ points to the center. We now look at motion along other curves, to separate change in the speed $|\mathbf{v}|$ from change in the direction $\mathbf{T}$.

The direction of motion is $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$. It depends on the path but not the speed (because we divide by $|\mathbf{v}|$). For turning we measure two things:

1. **How fast T turns: this will be the *curvature* $\kappa$ (kappa).**
2. **Which direction T turns: this will be the *normal vector* N.**

$\kappa$ and $\mathbf{N}$ depend, like $s$ and $\mathbf{T}$, only on the shape of the curve. Replacing $t$ by $2t$ or $t^2$ leaves them unchanged. For a circle we give the answers in advance. The normal vector $\mathbf{N}$ points to the center. The curvature $\kappa$ is 1/radius.

A smaller turning circle means a larger curvature $\kappa$: more bending.

The curvature $\kappa$ is ***change in direction*** $|d\mathbf{T}|$ ***divided by change in position*** $|ds|$. There are three formulas for $\kappa$—a direct one for graphs $y(x)$, a brutal but valuable one for any parametric curve $(x(t), y(t))$, and a neat formula that uses tbe vectors $\mathbf{v}$ and $\mathbf{a}$. We begin with the definition and the neat formula.

$$\textbf{DEFINITION} \quad \kappa = |d\mathbf{T}/ds| \qquad \textbf{FORMULA} \quad \kappa = |\mathbf{v} \times \mathbf{a}|/|\mathbf{v}|^3 \tag{1}$$

The definition does not involve the parameter $t$—but the calculations do. The position vector $\mathbf{R}(t)$ yields $\mathbf{v} = d\mathbf{R}/dt$ and $\mathbf{a} = d\mathbf{v}/dt$. If $t$ is changed to $2t$, the velocity $\mathbf{v}$ is doubled and $\mathbf{a}$ is multiplied by 4. Then $|\mathbf{v} \times \mathbf{a}|$ and $|\mathbf{v}|^3$ are multiplied by 8, and their ratio $\kappa$ is unchanged.

Proof of formula (1) Start from $\mathbf{v} = |\mathbf{v}|\mathbf{T}$ and compute its derivative $\mathbf{a}$:

$$\mathbf{a} = \frac{d|\mathbf{v}|}{dt}\mathbf{T} + |\mathbf{v}|\frac{d\mathbf{T}}{dt} \text{ by the product rule.}$$

Now take the cross product with $\mathbf{v} = |\mathbf{v}|\mathbf{T}$. Remember that $\mathbf{T} \times \mathbf{T} = \mathbf{0}$:

$$\mathbf{v} \times \mathbf{a} = |\mathbf{v}|\mathbf{T} \times |\mathbf{v}|\frac{d\mathbf{T}}{dt}. \tag{2}$$

We know that $|\mathbf{T}| = 1$. Equation (4) will show that $\mathbf{T}$ is perpendicular to $d\mathbf{T}/dt$. So $|\mathbf{v} \times \mathbf{a}|$ is the first length $|\mathbf{v}|$ times the second length $|\mathbf{v}|\,|d\mathbf{T}/dt|$. The factor $\sin\theta$ in the length of a cross product is 1 from the 90° angle. In other words

$$\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^2} \quad \text{and} \quad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}. \tag{3}$$

The chain rule brings the extra $|ds/dt| = |\mathbf{v}|$ into the denominator.

Before any examples, we show that $d\mathbf{T}/dt$ is perpendicular to $\mathbf{T}$. The reason is that $\mathbf{T}$ is a unit vector. Differentiate both sides of $\mathbf{T} \cdot \mathbf{T} = 1$:

$$\frac{d\mathbf{T}}{dt} \cdot \mathbf{T} + \mathbf{T} \cdot \frac{d\mathbf{T}}{dt} = 0 \qquad \textit{or} \qquad 2\mathbf{T} \cdot \frac{d\mathbf{T}}{dt} = 0. \tag{4}$$

That proof used the product rule $\mathbf{U}'\cdot\mathbf{V}+\mathbf{U}\cdot\mathbf{V}'$ for the derivative of $\mathbf{U}\cdot\mathbf{V}$ (Problem 23, with $\mathbf{U}=\mathbf{V}=\mathbf{T}$). Think of the vector $\mathbf{T}$ moving around the unit sphere. To keep a constant length $(\mathbf{T}+d\mathbf{T})\cdot(\mathbf{T}+d\mathbf{T})=1$, we need $2\mathbf{T}\cdot d\mathbf{T}=0$. Movement $d\mathbf{T}$ is perpendicular to radius vector $\mathbf{T}$.

Our first examples will **be plane curves**. The position vector $\mathbf{R}(t)$ has components $x(t)$ and $y(t)$ but no $z(t)$. Look at the components of $\mathbf{v}$ and $\mathbf{a}$ and $\mathbf{v}\times\mathbf{a}$ ($x'$ means $dx/dt$):

$$\begin{array}{cccc} \mathbf{R} & x(t) & y(t) & 0 \\ \mathbf{v} & x'(t) & y'(t) & 0 \\ \mathbf{a} & x''(t) & y''(t) & 0 \\ \mathbf{v}\times\mathbf{a} & 0 & 0 & x'y''-y'x'' \end{array} \qquad |\mathbf{v}|=\sqrt{|x'|^2+|y'|^2} \qquad \kappa=\frac{|x'y''-y'x''|}{((x')^2+(y')^2)^{3/2}} \tag{5}$$

Equation (5) is the brutal but valuable formula for $\kappa$. Apply it to movement around a circle. We should find $\kappa=1/\text{radius } a$:

**EXAMPLE 1** When $x=a\cos\omega t$ and $y=a\sin\omega t$ we substitute $x'$, $y'$, $x''$, $y''$ into (5):

$$\kappa=\frac{(-\omega a\sin\omega t)(-\omega^2 a\sin\omega t)-(\omega a\cos\omega t)(-\omega^2 a\cos\omega t)}{[(\omega a\sin\omega t)^2+(\omega a\cos\omega t)^2]^{3/2}}=\frac{\omega^3a^2}{[\omega^2a^2]^{3/2}}.$$

This is $\omega^3a^2/\omega^3a^3$ and $\omega$ cancels. The speed makes no difference to $\kappa=1/a$.

The third formula for $\kappa$ applies to an ordinary plane curve given by $y(x)$. The parameter $t$ is $x$! You see the square root in the speed $|\mathbf{v}|=ds/dx$:

$$\begin{array}{cccc} \mathbf{R} & x & y(x) & 0 \\ \mathbf{v} & 1 & dy/dx & 0 \\ \mathbf{a} & 0 & d^2y/dx^2 & 0 \\ \mathbf{v}\times\mathbf{a} & 0 & 0 & d^2y/dx^2 \end{array} \qquad |\mathbf{v}|=\sqrt{1+(dy/dx)^2} \qquad \kappa=\frac{|d^2y/dx^2|}{(1+(dy/dx)^2)^{3/2}} \tag{6}$$

In practice this is the most popular formula for $\kappa$. The most popular approximation is $|d^2y/dx^2|$. (The denominator is omitted.) For the bending of a beam, the nonlinear equation uses $\kappa$ and the linear equation uses $d^2y/dx^2$. We can see the difference for a parabola:

**EXAMPLE 2** The curvature of $y=\frac{1}{2}x^2$ is $\kappa=|y''|/(1+(y')^2)^{3/2}=1/(1+x^2)^{3/2}$.

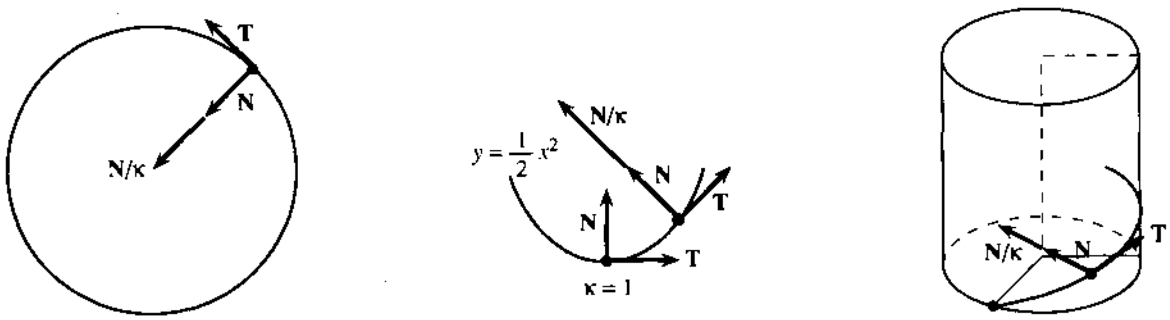



**Fig. 12.7** Normal $\mathbf{N}$ divided by curvature $\kappa$ for circle and parabola and unit helix.

The approximation is $y'' = 1$. This agrees with $\kappa$ at $x = 0$, where the parabola turns the corner. But for large $x$, the curvature approaches zero. Far out on the parabola, we go a long way for a small change in direction.

The parabola $y = -\frac{1}{2}x^2$, opening down, has the same $\kappa$. Now try a space curve.

**EXAMPLE 3** Find the curvature of the unit helix $\mathbf{R} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$.

Take the cross product of $\mathbf{v} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$ and $\mathbf{a} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$:

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{k}.$$

This cross product has length $\sqrt{2}$. Also the speed is $|\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$:

$$\kappa = |\mathbf{v} \times \mathbf{a}|/|\mathbf{v}|^3 = \sqrt{2}/(\sqrt{2})^3 = \tfrac{1}{2}.$$

Compare with a unit circle. Without the climbing term $t\mathbf{k}$, the curvature would be 1. Because of climbing, each turn of the helix is longer and $\kappa = \frac{1}{2}$.

That makes one think: Is the helix twice as long as the circle? No. The length of a turn is only increased by $|\mathbf{v}| = \sqrt{2}$. The other $\sqrt{2}$ is because the tangent $\mathbf{T}$ slopes upward. The shadow in the base turns a full 360°, but $\mathbf{T}$ turns less.

## THE NORMAL VECTOR N

The discussion is bringing us to an important vector. Where $\kappa$ measures the ***rate*** of turning, the unit vector $\mathbf{N}$ gives the ***direction*** of turning. $\mathbf{N}$ is perpendicular to $\mathbf{T}$, and in the plane that leaves practically no choice. Turn left or right. For a space curve, follow $d\mathbf{T}$. Remember equation (4), which makes $d\mathbf{T}$ perpendicular to $\mathbf{T}$.

***The normal vector* $\mathbf{N}$ *is a unit vector along* $d\mathbf{T}/dt$. *It is perpendicular to* $\mathbf{T}$:**

$$\textbf{DEFINITION}\quad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \qquad \textbf{FORMULA}\quad \mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}. \tag{7}$$

**EXAMPLE 4** Find the normal vector $\mathbf{N}$ for the same helix $\mathbf{R} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$.

Solution Copy $\mathbf{v}$ from Example 3, divide by $|\mathbf{v}|$, and compute $d\mathbf{T}/dt$:

$$\mathbf{T} = \mathbf{v}/|\mathbf{v}| = (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})/\sqrt{2} \quad \text{and} \quad d\mathbf{T}/dt = (-\cos t\,\mathbf{i} - \sin t\,\mathbf{j})/\sqrt{2}.$$

To change $d\mathbf{T}/dt$ into a unit vector, cancel the $\sqrt{2}$. ***The normal vector is*** $\mathbf{N} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$. It is perpendicular to $\mathbf{T}$. Since the $\mathbf{k}$ component is zero, $\mathbf{N}$ is horizontal. The tangent $\mathbf{T}$ slopes up at 45°—it goes around the circle at that latitude. The normal $\mathbf{N}$ is tangent to this circle ($\mathbf{N}$ is tangent to the path of the tangent!). So $\mathbf{N}$ stays horizontal as the helix climbs.

There is also a third direction, perpendicular to $\mathbf{T}$ and $\mathbf{N}$. It is the ***binormal*** vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, computed in Problems 25–30. The unit vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ provide the natural coordinate system for the path—along the curve, in the plane of the curve, and out of that plane. The theory is beautiful but the computations are not often done—we stop here.

## TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

May I return a last time to the gas pedal and brake and steering wheel? The first two give acceleration along **T**. Turning gives acceleration along **N**. The rate of turning (curvature $\kappa$) and the direction **N** are established. We now ask about the *force* required. Newton's Law is $\mathbf{F} = m\mathbf{a}$, so we need the acceleration **a**—especially its component along **T** and its component along **N**.

$$\textit{The acceleration is } \mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left[\frac{ds}{dt}\right]^2\mathbf{N}. \tag{8}$$

For a straight path, $d^2s/dt^2$ is the only acceleration—the ordinary second derivative. The term $\kappa(ds/dt)^2$ is the acceleration in turning. Both have the dimension of length/(time)$^2$.

The force to steer around a corner depends on curvature and speed—as all drivers know. Acceleration is the derivative of $\mathbf{v} = |\mathbf{v}|\mathbf{T} = (ds/dt)\mathbf{T}$:

$$\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{ds}\frac{ds}{dt}. \tag{9}$$

That last term is $\kappa(ds/dt)^2\mathbf{N}$, since $d\mathbf{T}/ds = \kappa\mathbf{N}$ by formula (7). So (8) is proved.

**EXAMPLE 5** A fixed speed $ds/dt = 1$ gives $d^2s/dt^2 = 0$. The only acceleration is $\kappa\mathbf{N}$.

**EXAMPLE 6** Find the components of **a** for circular speed-up $\mathbf{R}(t) = \cos t^2\,\mathbf{i} + \sin t^2\,\mathbf{j}$.

Without stopping to think, compute $d\mathbf{R}/dt = \mathbf{v}$ and $ds/dt = |\mathbf{v}|$ and $\mathbf{v}/|\mathbf{v}| = \mathbf{T}$:

$$\mathbf{v} = -2t\sin t^2\,\mathbf{i} + 2t\cos t^2\,\mathbf{j}, \quad |\mathbf{v}| = 2t, \quad \mathbf{T} = -\sin t^2\,\mathbf{i} + \cos t^2\,\mathbf{j}.$$

The derivative of $ds/dt = |\mathbf{v}|$ is $d^2s/dt^2 = 2$. The derivative of **v** is **a**:

$$\mathbf{a} = -2\sin t^2\,\mathbf{i} + 2\cos t^2\,\mathbf{j} - 4t^2\cos t^2\,\mathbf{i} - 4t^2\sin t^2\,\mathbf{j}.$$

In the first terms of **a** we see $2\mathbf{T}$. In the last terms we must be seeing $\kappa|\mathbf{v}|^2\mathbf{N}$. Certainly $|\mathbf{v}|^2 = 4t^2$ and $\kappa = 1$, because the circle has radius 1. Thus $\mathbf{a} = 2\mathbf{T} + 4t^2\mathbf{N}$ has the tangential component 2 and normal component $4t^2$—acceleration along the circle and in to the center.

***Table of Formulas***

$\mathbf{v} = d\mathbf{R}/dt \quad \mathbf{a} = d\mathbf{v}/dt$

$|\mathbf{v}| = ds/dt \quad \mathbf{T} = \mathbf{v}/|\mathbf{v}| = |d\mathbf{R}/ds|$

Curvature $\kappa = |d\mathbf{T}/ds| = |\mathbf{v} \times \mathbf{a}|/|\mathbf{v}|^3$

Plane curves $\kappa = \dfrac{|x'y'' - y'x''|}{((x')^2 + (y')^2)^{3/2}} = \dfrac{|d^2y/dx^2|}{(1 + (dy/dx)^2)^{3/2}}$

Normal vector $\mathbf{N} = \dfrac{1}{\kappa}\dfrac{d\mathbf{T}}{ds} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

Acceleration $\mathbf{a} = (d^2s/dt^2)\mathbf{T} + \kappa|\mathbf{v}|^2\mathbf{N}$

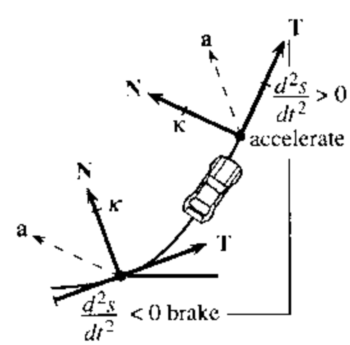



**Fig. 12.8** Components of **a** as car turns corner.

## 12.3 EXERCISES

**Read-through questions**

The curvature tells how fast the curve __a__. For a circle of radius $a$, the direction changes by $2\pi$ in a distance __b__, so $\kappa =$ __c__. For a plane curve $y=f(x)$ the formula is $\kappa = |y''|/$ __d__. The curvature of $y=\sin x$ is __e__. At a point where $y''=0$ (an __f__ point) the curve is momentarily straight and $\kappa =$ __g__. For a space curve $\kappa = |\mathbf{v}\times\mathbf{a}|/$ __h__.

The normal vector $\mathbf{N}$ is perpendicular to __i__. It is a __j__ vector along the derivative of $\mathbf{T}$, so $\mathbf{N} =$ __k__. For motion around a circle $\mathbf{N}$ points __l__. Up a helix $\mathbf{N}$ also points __m__. Moving at unit speed on any curve, the time $t$ is the same as the __n__ $s$. Then $|\mathbf{v}| =$ __o__ and $d^2s/dt^2 =$ __p__ and $\mathbf{a}$ is in the direction of __q__.

Acceleration equals __r__ $\mathbf{T}+$ __s__ $\mathbf{N}$. At unit speed around a unit circle, those components are __t__. An astronaut who spins once a second in a radius of one meter has $|\mathbf{a}| =$ __u__ meters/sec², which is about __v__ $g$.

**Compute the curvature $\kappa$ in Problems 1–8.**

1 $y=e^x$

2 $y=\ln x$ (where is $\kappa$ largest?)

3 $x=2\cos t$, $y=2\sin t$

4 $x=\cos t^2$, $y=\sin t^2$

5 $x=1+t^2$, $y=3t^2$ (the path is a ________).

6 $x=\cos^3 t$, $y=\sin^3 t$

7 $r=\theta=t$ (so $x=t\cos t$, $y=$ ________)

8 $x=t$, $y=\ln\cos t$

9 Find $\mathbf{T}$ and $\mathbf{N}$ in Problem 4.

10 Show that $\mathbf{N}=\sin t\,\mathbf{i}+\cos t\,\mathbf{j}$ in Problem 6.

11 Compute $\mathbf{T}$ and $\mathbf{N}$ in Problem 8.

12 Find the speed $|\mathbf{v}|$ and curvature $\kappa$ of a projectile:

$$x=(v_0\cos\alpha)t,\ y=(v_0\sin\alpha)t-\tfrac{1}{2}gt^2.$$

13 Find $\mathbf{T}$ and $|\mathbf{v}|$ and $\kappa$ for the helix $\mathbf{R}=3\cos t\,\mathbf{i}+3\sin t\,\mathbf{j}+4t\,\mathbf{k}$. How much longer is a turn of the helix than the corresponding circle? What is the upward slope of $\mathbf{T}$?

14 When $\kappa=0$ the path is a ________. This happens when $\mathbf{v}$ and $\mathbf{a}$ are ________. Then $\mathbf{v}\times\mathbf{a}=$ ________.

15 Find the curvature of a cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$.

16 If all points of a curve are moved twice as far from the origin ($x\to 2x$, $y\to 2y$), what happens to $\kappa$? What happens to $\mathbf{N}$?

17 Find $\kappa$ and $\mathbf{N}$ at $\theta=\pi$ for the hypocycloid $x=4\cos\theta+\cos 4\theta$, $y=4\sin\theta-\sin 4\theta$.

18 From $\mathbf{v}=|\mathbf{v}|\mathbf{T}$ and $\mathbf{a}$ in equation (8), derive $\kappa=|\mathbf{v}\times\mathbf{a}|/|\mathbf{v}|^3$.

19 From a point on the curve, go along the vector $\mathbf{N}/\kappa$ to find the ***center of curvature***. Locate this center for the point $(1,0)$ on the circle $x=\cos t$, $y=\sin t$ and the ellipse $x=\cos t$, $y=2\sin t$ and the parabola $y=\frac{1}{2}(x^2-1)$. The path of the center of curvature is the "*evolute*" of the curve.

20 Which of these depend only on the shape of the curve, and which depend also on the speed? $\mathbf{v}$, $\mathbf{T}$, $|\mathbf{v}|$, $s$, $\kappa$, $\mathbf{a}$, $\mathbf{N}$, $\mathbf{B}$.

21 A plane curve through $(0,0)$ and $(2,0)$ with constant curvature $\kappa$ is the circular arc ________. For which $\kappa$ is there no such curve?

22 Sketch a smooth curve going through $(0,0)$, $(1,-1)$, and $(2,0)$. Somewhere $d^2y/dx^2$ is at least ________. Somewhere the curvature is at least ________. (Proof is for instructors only.)

23 For plane vectors, the ordinary product rule applied to $U_1V_1+U_2V_2$ shows that $(\mathbf{U}\cdot\mathbf{V})'=\mathbf{U}'\cdot\mathbf{V}+$ ________.

24 If $\mathbf{v}$ is perpendicular to $\mathbf{a}$, prove that the speed is constant. True or false: The path is a circle.

**Problems 25–30 work with the T-N-B system—along the curve, in the plane of the curve, perpendicular to that plane.**

25 Compute $\mathbf{B}=\mathbf{T}\times\mathbf{N}$ for the helix $\mathbf{R}=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+t\mathbf{k}$ in Examples 3–4.

26 Using Problem 23, differentiate $\mathbf{B}\cdot\mathbf{T}=0$ and $\mathbf{B}\cdot\mathbf{B}=1$ to show that $\mathbf{B}'$ is perpendicular to $\mathbf{T}$ and $\mathbf{B}$. So $d\mathbf{B}/ds=-\tau\mathbf{N}$ for some number $\tau$ called the ***torsion***.

27 Compute the torsion $\tau=|d\mathbf{B}/ds|$ for the helix in Problem 25.

28 Find $\mathbf{B}=\mathbf{T}\times\mathbf{N}$ for the curve $x=1$, $y=t$, $z=t^2$.

29 A circle lies in the $xy$ plane. Its normal $\mathbf{N}$ lies ________ and $\mathbf{B}=$ ________ and $\tau=|d\mathbf{B}/ds|=$ ________.

30 The Serret-Frenet formulas are $d\mathbf{T}/ds=\kappa\mathbf{N}$, $d\mathbf{N}/ds=-\kappa\mathbf{T}+\tau\mathbf{B}$, $d\mathbf{B}/ds=-\tau\mathbf{N}$. We know the first and third. Differentiate $\mathbf{N}=-\mathbf{T}\times\mathbf{B}$ to find the second.

31 The angle $\theta$ from the $x$ axis to the tangent line is $\theta=\tan^{-1}(dy/dx)$, when $dy/dx$ is the slope of the curve.

(a) Compute $d\theta/dx$.

(b) Divide by $ds/dx=(1+(dy/dx)^2)^{1/2}$ to show that $|d\theta/ds|$ is $\kappa$ in equation (5). Curvature is change in direction $|d\theta|$ divided by change in position $|ds|$.

32 If the tangent direction is at angle $\theta$ then $\mathbf{T}=\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}$. In Problem 31 $|d\theta/ds|$ agreed with $\kappa=|d\mathbf{T}/ds|$ because $|d\mathbf{T}/d\theta|=$ ________.

**In 33–37 find the T and N components of acceleration.**

**33** $x = 5\cos\omega t,\ y = 5\sin\omega t,\ z = 0$ (circle)

**34** $x = 1 + t,\ y = 1 + 2t,\ z = 1 + 3t$ (line)

**35** $x = t\cos t,\ y = t\sin t,\ z = 0$ (spiral)

**36** $x = e^t\cos t,\ y = e^t\sin t,\ z = 0$ (spiral)

**37** $x = 1,\ y = t,\ z = t^2$.

**38** For the spiral in 36, show that the angle between **R** and **a** (position and acceleration) is constant. Find the angle.

**39** Find the curvature of a polar curve $r = F(\theta)$.

## 12.4 Polar Coordinates and Planetary Motion

This section has a general purpose—to do vector calculus in ***polar coordinates***. It also has a specific purpose—to study ***central forces*** and the ***motion of planets***. The main gravitational force on a planet is from the sun. It is a central force, because it comes from the sun at the center. Polar coordinates are natural, so the two purposes go together.

You may feel that the planets are too old for this course. But Kepler's laws are more than theorems, they are something special in the history of mankind—"the greatest scientific discovery of all time." If we can recapture that glory we should do it. Part of the greatness is in the difficulty—Kepler was working sixty years before Newton discovered calculus. From pages of observations, and some terrific guesses, a theory was born. We will try to preserve the greatness without the difficulty, and show how elliptic orbits come from calculus. The first conclusion is quick.

***Motion in a central force field always stays in a plane.***

F is a multiple of the vector **R** from the origin (central force). **F** also equals $m\mathbf{a}$ (Newton's Law). Therefore **R** and **a** are in the same direction and $\mathbf{R}\times\mathbf{a} = \mathbf{0}$. Then $\mathbf{R}\times\mathbf{v}$ has zero derivative and is constant:

$$\text{by the product rule:}\quad \frac{d}{dt}(\mathbf{R}\times\mathbf{v}) = \mathbf{v}\times\mathbf{v} + \mathbf{R}\times\mathbf{a} = \mathbf{0} + \mathbf{0}. \tag{1}$$

$\mathbf{R}\times\mathbf{v}$ is a constant vector **H**. So **R** stays in the plane perpendicular to **H**.

How does a planet move in that plane? We turn to polar coordinates. At each point except the origin (where the sun is), $\mathbf{u}_r$ ***is the unit vector pointing outward.*** It is the position vector **R** divided by its length $r$ (which is $\sqrt{x^2+y^2}$):

$$\mathbf{u}_r = \mathbf{R}/r = (x\mathbf{i} + y\mathbf{j})/r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}. \tag{2}$$

That is a unit vector because $\cos^2\theta + \sin^2\theta = 1$. It goes out from the center. Figure 12.9 shows $\mathbf{u}_r$ and the second unit vector $\mathbf{u}_\theta$ at a 90° angle:

$$\mathbf{u}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}. \tag{3}$$

The dot product is $\mathbf{u}_r\cdot\mathbf{u}_\theta = 0$. The subscripts $r$ and $\theta$ indicate direction (not derivative).

**Question 1:** How do $\mathbf{u}_r$ and $\mathbf{u}_\theta$ change as $r$ changes (out a ray)? *They don't.*

**Question 2:** How do $\mathbf{u}_r$ and $\mathbf{u}_\theta$ change as $\theta$ changes? *Take the derivative:*

$$\begin{aligned} d\mathbf{u}_r/d\theta &= -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} = \mathbf{u}_\theta \\ d\mathbf{u}_\theta/d\theta &= -\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j} = -\mathbf{u}_r. \end{aligned} \tag{4}$$

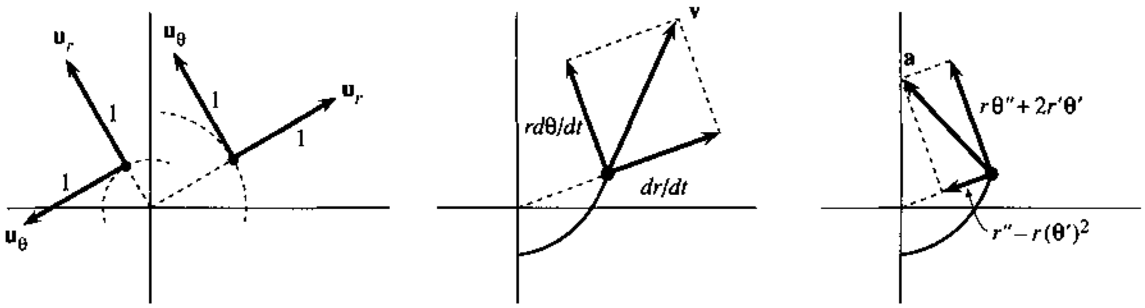



**Fig. 12.9** $\mathbf{u}_r$ is outward, $\mathbf{u}_\theta$ is around the center. Components of $\mathbf{v}$ and $\mathbf{a}$ in those directions.

Since $\mathbf{u}_r = \mathbf{R}/r$, one formula is simple: ***The position vector is*** $\mathbf{R} = r\mathbf{u}_r$. For its derivative $\mathbf{v} = d\mathbf{R}/dt$, use the chain rule $d\mathbf{u}_r/dt = (d\mathbf{u}_r/d\theta)(d\theta/dt) = (d\theta/dt)\mathbf{u}_\theta$:

$$\textbf{\textit{The velocity is}}\ \mathbf{v} = \frac{d}{dt}(r\mathbf{u}_r) = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta. \tag{5}$$

The outward speed is $dr/dt$. The circular speed is $r\,d\theta/dt$. The sum of squares is $|\mathbf{v}|^2$.

Return one more time to steady motion around a circle, say $r = 3$ and $\theta = 2t$. The velocity is $\mathbf{v} = 6\mathbf{u}_\theta$, all circular. The acceleration is $-12\mathbf{u}_r$, all inward. For circles $\mathbf{u}_\theta$ is the tangent vector $\mathbf{T}$. But the unit vector $\mathbf{u}_r$ points outward and $\mathbf{N}$ points inward—the way the curve turns.

Now we tackle acceleration for any motion in polar coordinates. There can be speedup in $r$ and speedup in $\theta$ (also change of direction). Differentiate $\mathbf{v}$ in (5) by the product rule:

$$\frac{d\mathbf{v}}{dt} = \frac{d^2r}{dt^2}\mathbf{u}_r + \frac{dr}{dt}\frac{d\mathbf{u}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + r\frac{d\theta}{dt}\frac{d\mathbf{u}_\theta}{dt}.$$

For $d\mathbf{u}_r/dt$ and $d\mathbf{u}_\theta/dt$, multiply equation (4) by $d\theta/dt$. Then all terms contain $\mathbf{u}_r$ or $\mathbf{u}_\theta$. The formula for $\mathbf{a}$ is famous but not popular (except it got us to the moon):

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{u}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{u}_\theta. \tag{6}$$

In the steady motion with $r = 3$ and $\theta = 2t$, only one acceleration term is nonzero: $\mathbf{a} = -12\mathbf{u}_r$. Formula (6) can be memorized (maybe). Problem 14 gives a new way to reach it, using $re^{i\theta}$.

**EXAMPLE 1** Find $\mathbf{R}$ and $\mathbf{v}$ and $\mathbf{a}$ for speedup $\theta = t^2$ around the circle $r = 1$.

Solution The position vector is $\mathbf{R} = \mathbf{u}_r$. Then $\mathbf{v}$ and $\mathbf{a}$ come from (5–6):

$$\mathbf{v} = (r\,d\theta/dt)\mathbf{u}_\theta = 2t\mathbf{u}_\theta \qquad \mathbf{a} = -(2t)^2\mathbf{u}_r + 2\mathbf{u}_\theta.$$

This question and answer were also in Example 6 of the previous section. The acceleration was $2\mathbf{T} + 4t^2\mathbf{N}$. Notice again that $\mathbf{T} = \mathbf{u}_\theta$ and $\mathbf{N} = -\mathbf{u}_r$, going round the circle.

**EXAMPLE 2** Find $\mathbf{R}$ and $\mathbf{v}$ and $|\mathbf{v}|$ and $\mathbf{a}$ for the spiral motion $r = 3t$, $\theta = 2t$.

Solution The position vector is $\mathbf{R} = 3t\,\mathbf{u}_r$. Equation (5) gives velocity and speed:

$$\mathbf{v} = 3\mathbf{u}_r + 6t\mathbf{u}_\theta \qquad \text{and} \qquad |\mathbf{v}| = \sqrt{(3)^2 + (6t)^2}.$$

The motion goes *out* and also *around*. From (6) the acceleration is $-12t\,\mathbf{u}_r + 12\mathbf{u}_\theta$. The same answers would come more slowly from $\mathbf{R} = 3t\cos 2t\,\mathbf{i} + 3t\sin 2t\,\mathbf{j}$.

This example uses polar coordinates, but *the motion is not circular*. One of Kepler's inspirations, after many struggles, was to get away from circles.

## KEPLER'S LAWS

You may know that before Newton and Leibniz and calculus and polar coordinates, Johannes Kepler discovered three laws of planetary motion. He was the court mathematician to the Holy Roman Emperor, who mostly wanted predictions of wars. Kepler also determined the date of every Easter—no small problem. His triumph was to discover patterns in the observations made by astronomers (especially by Tycho Brahe). Galileo and Copernicus expected circles, but Kepler found ellipses.

***Law 1:*** Each planet travels in an ellipse with one focus at the sun.

***Law 2:*** The vector from sun to planet sweeps out area at a steady rate: $dA/dt =$ constant.

***Law 3:*** The length of the planet's year is $T = ka^{3/2}$, where $a =$ maximum distance from the center (not the sun) and $k = 2\pi/\sqrt{GM}$ is the same for all planets.

With calculus the proof of these laws is a thousand times quicker. But Law 2 is the only easy one. The sun exerts a central force. Equation (1) gave $\mathbf{R} \times \mathbf{v} = \mathbf{H} =$ constant for central forces. Replace $\mathbf{R}$ by $r\mathbf{u}_r$ and replace $\mathbf{v}$ by equation (5):

$$\mathbf{H} = r\mathbf{u}_r \times \left(\frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta\right) = r^2\frac{d\theta}{dt}(\mathbf{u}_r \times \mathbf{u}_\theta). \tag{7}$$

This vector $\mathbf{H}$ is constant, so ***its length $h = r^2 d\theta/dt$ is constant.*** In polar coordinates, the area is $dA = \frac{1}{2}r^2 d\theta$. This area $dA$ is swept out by the planct (Figure 12.10), and we have proved Law 2:

$$dA/dt = \tfrac{1}{2}r^2\,d\theta/dt = \tfrac{1}{2}h = \textbf{\textit{constant}}. \tag{8}$$

Near the sun $r$ is small. So $d\theta/dt$ is big and planets go around faster.

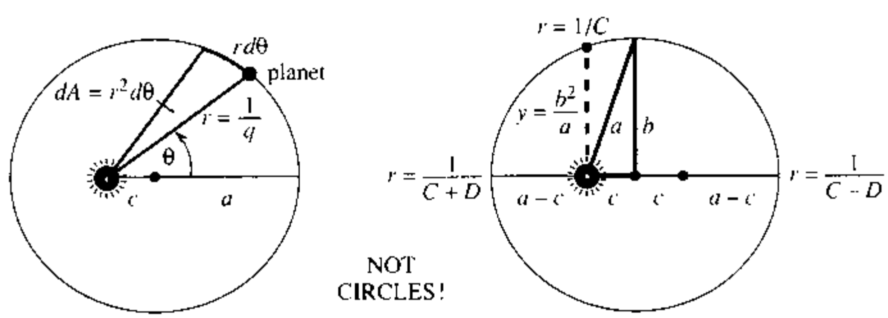



**Fig. 12.10** The planet is on an ellipse with the sun at a focus. Note $a$, $b$, $c$, $q$.

Now for Law 1, about ellipses. We are aiming for $1/r = C - D\cos\theta$, which is *the polar coordinate equation of an ellipse*. It is easier to write $q$ than $1/r$, and find an equation for $q$. The equation we will reach is $d^2q/d\theta^2 + q = C$. The desired $q = C - D\cos\theta$ solves that equation (check this), and gives us Kepler's ellipse.

The first step is to connect $dr/dt$ to $dq/d\theta$ by the chain rule:

$$\frac{dr}{dt}=\frac{d}{dt}\left(\frac{1}{q}\right)=\frac{-1}{q^2}\frac{dq}{dt}=\frac{-1}{q^2}\frac{dq}{d\theta}\frac{d\theta}{dt}=-h\frac{dq}{d\theta}. \tag{9}$$

Notice especially $d\theta/dt=h/r^2=hq^2$. What we really want are second derivatives:

$$\frac{d^2r}{dt^2}=-h\frac{d}{dt}\left(\frac{dq}{d\theta}\right)=-h\frac{d}{d\theta}\left(\frac{dq}{d\theta}\right)\frac{d\theta}{dt}=-h^2q^2\frac{d^2q}{d\theta^2}. \tag{10}$$

After this trick of introducing $q$, we are ready for physics. The planet obeys Newton's Law $\mathbf{F}=m\mathbf{a}$, and the central force $\mathbf{F}$ is the sun's gravity:

$$\frac{\mathbf{F}}{m}=\mathbf{a} \quad \text{is} \quad -\frac{GM}{r^2}=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2. \tag{11}$$

That right side is the $\mathbf{u}_r$ component of $\mathbf{a}$ in (6). Change $r$ to $1/q$ and change $d\theta/dt$ to $hq^2$. The preparation in (10) allows us to rewrite $d^2r/dt^2$ in equation (11). That equation becomes

$$-GM\,q^2=-h^2q^2\frac{d^2q}{d\theta^2}-\frac{1}{q}(hq^2)^2.$$

Dividing by $-h^2q^2$ gives what we hoped for—the simple equation for $q$:

$$d^2q/d\theta^2+q=GM/h^2=C\ (\textit{a constant}). \tag{12}$$

***The solution is*** $q=C-D\cos\theta$. Section 9.3 gave this polar equation for an ellipse or parabola or hyperbola. To be sure it is an ellipse, an astronomer computes $C$ and $D$ from the sun's mass $M$ and the constant $G$ and the earth's position and velocity. *The main point is that* $C>D$. Then $q$ is never zero and $r$ is never infinite. Hyperbolas and parabolas are ruled out, and the orbit in Figure 12.10 must be an ellipse.†

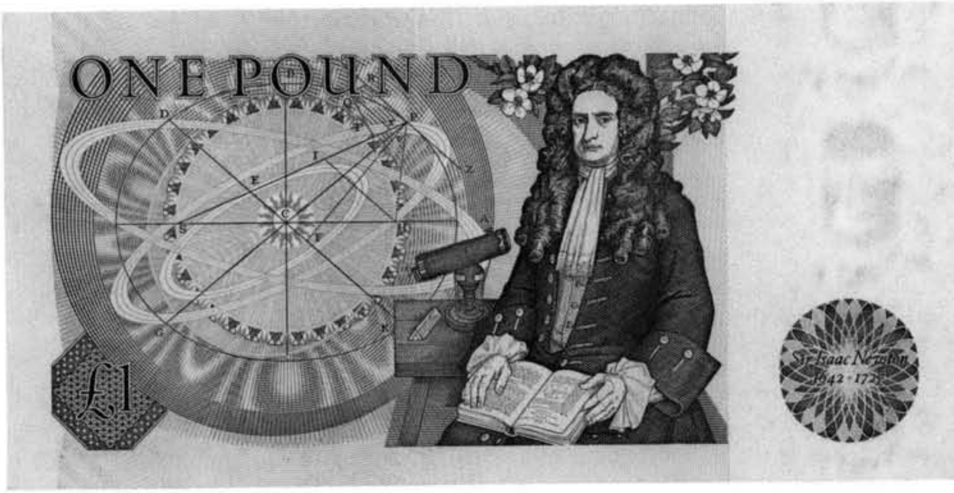
Astronomy is really impressive. You should visit the Greenwich Observatory in London, to see how Halley watched his comet. He amazed the world by predicting the day it would return. Also the discovery of Neptune was pure mathematics—the path of Uranus was not accounted for by the sun and known planets. LeVerrier computed a point in the sky and asked a Berlin astronomer to look. Sure enough Neptune was there.

Recently one more problem was solved—to explain the gap in the asteroids around Jupiter. The reason is "*chaos*"—the three-body problem goes unstable and an asteroid won't stay in that orbit. We have come a long way from circles.

***Department of Royal Mistakes*** The last pound note issued by the Royal Mint showed Newton looking up from his great book *Principia Mathematica.* He is not smiling and we can see why. The artist put the sun at the center! Newton has just proved it is at the focus. True, the focus is marked $S$ and the planet is $P$. But those rays at the center brought untold headaches to the Mint—the note is out of circulation. I gave an antique dealer three pounds for it (in coins).

***Kepler's third law*** gives the time $T$ to go around the ellipse—the planet's year. What is special in the formula is $a^{3/2}$—and for Kepler himself, the 15th of May 1618 was unforgettable: "the right ratio outfought the darkness of my mind, by the great proof afforded by my labor of seventeen years on Brahe's observations." The second

†An amateur sees the planet come around again, and votes for an ellipse.




law $dA/dt = \frac{1}{2}h$ is the key, plus two facts about an ellipse—its area $\pi ab$ and the height $b^2/a$ above the sun:

**1.** The area $A = \int_0^T \frac{dA}{dt}\,dt = \frac{1}{2}hT$ must equal $\pi ab$, so $T = \frac{2\pi ab}{h}$

**2.** The distance $r = 1/C$ at $\theta = \pi/2$ must equal $b^2/a$, so $b = \sqrt{a/C}$.

The height $b^2/a$ is in Figure 12.10 and Problems 25–26. The constant $C = GM/h^2$ is in equation (12). Put them together to find the period:

$$T = \frac{2\pi ab}{h} = \frac{2\pi a}{h}\sqrt{\frac{a}{C}} = \frac{2\pi}{\sqrt{GM}}a^{3/2}. \tag{13}$$

To think of Kepler guessing $a^{3/2}$ is amazing. To think of Newton proving Kepler's laws by calculus is also wonderful—because we can do it too.

**EXAMPLE 3** When a satellite goes around in a circle, find the time $T$.

Let $r$ be the radius and $\omega$ be the angular velocity. The time for a complete circle (angle $2\pi$) is $T = 2\pi/\omega$. The acceleration is $GM/r^2$ from gravity, and it is also $r\omega^2$ for circular motion. Therefore Kepler is proved right:

$$r\omega^2 = GM/r^2 \quad\Rightarrow\quad \omega = \sqrt{GM/r^3} \quad\Rightarrow\quad T = 2\pi/\omega = 2\pi r^{3/2}/\sqrt{GM}.$$

## 12.4 EXERCISES

**Read-through questions**

A central force points toward __a__. Then $\mathbf{R} \times d^2\mathbf{R}/dt^2 = \mathbf{0}$ because __b__. Therefore $\mathbf{R} \times d\mathbf{R}/dt$ is a __c__ (called $\mathbf{H}$).

In polar coordinates, the outward unit vector is $\mathbf{u}_r = \cos\theta\,\mathbf{i} +$ __d__. Rotated by 90° this becomes $\mathbf{u}_\theta =$ __e__. The position vector $\mathbf{R}$ is the distance $r$ times __f__. The velocity $\mathbf{v} = d\mathbf{R}/dt$ is __g__ $\mathbf{u}_r +$ __h__ $\mathbf{u}_\theta$. For steady motion around the circle $r = 5$ with $\theta = 4t$, $\mathbf{v}$ is __i__ and $|\mathbf{v}|$ is __j__ and $\mathbf{a}$ is __k__.

For motion under a circular force, $r^2$ times __l__ is constant. Dividing by 2 gives Kepler's second law $dA/dt =$ __m__. The first law says that the orbit is an __n__ with the sun at __o__. The polar equation for a conic section is __p__ $= C - D\cos\theta$. Using $\mathbf{F} = m\mathbf{a}$ we found $q_{\theta\theta} +$ __q__ $= C$. So the path is a conic section; it must be an ellipse because __r__. The properties of an ellipse lead to the period $T =$ __s__, which is Kepler's third law.

1 Find the unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$ at the point $(0, 2)$. The $\mathbf{u}_r$ and $\mathbf{u}_\theta$ components of $\mathbf{v} = \mathbf{i} + \mathbf{j}$ at that point are ________.

2 Find $\mathbf{u}_r$ and $\mathbf{u}_\theta$ at $(3, 3)$. If $\mathbf{v} = \mathbf{i} + \mathbf{j}$ then $\mathbf{v} =$ ________ $\mathbf{u}_r$. Equation (5) gives $dr/dt =$ ________ and $d\theta/dt =$ ________.

3 At the point $(1, 2)$, velocities in the direction ________ will give $dr/dt = 0$. Velocities in the direction ________ will give $d\theta/dt = 0$.

4 Traveling on the cardioid $r = 1 - \cos\theta$ with $d\theta/dt = 2$, what is $\mathbf{v}$? How long to go around the cardioid (no integration involved)?

5 If $r = e^\theta$ and $\theta = 3t$, find $\mathbf{v}$ and $\mathbf{a}$ when $t = 1$.

6 If $r = 1$ and $\theta = \sin t$, describe the path and find $\mathbf{v}$ and $\mathbf{a}$ from equations (5–6). Where is the velocity zero?

7 (important) $\mathbf{R} = 4\cos 5t\,\mathbf{i} + 4\sin 5t\,\mathbf{j} = 4\mathbf{u}_r$ travels on a circle of radius 4 with $\theta = 5t$ and speed 20. Find the components of $\mathbf{v}$ and $\mathbf{a}$ in three systems: $\mathbf{i}$ and $\mathbf{j}$, $\mathbf{T}$ and $\mathbf{N}$, $u_r$ and $\mathbf{u}_\theta$.

8 When is the circle $r = 4$ completed, if the speed is $8t$? Find $\mathbf{v}$ and $\mathbf{a}$ at the return to the starting point $(4, 0)$.

9 The $\mathbf{u}_\theta$ component of acceleration is ________ $= 0$ for a central force, which is in the direction of ________. Then $r^2 d\theta/dt$ is constant (new proof) because its derivative is $r$ times ________.

10 If $r^2 d\theta/dt = 2$ for travel up the line $x = 1$, draw a triangle to show that $r = \sec\theta$ and integrate to find the time to reach $(1, 1)$.

11 A satellite is $r = 10{,}000$ km from the center of the Earth, traveling perpendicular to the radius vector at 4 km/sec. Find $d\theta/dt$ and $h$.

12 From $|\mathbf{u}_r| = 1$, it follows that $d\mathbf{u}_r/dr$ and $d\mathbf{u}_r/d\theta$ are ________ to $\mathbf{u}_r$ (Section 12.3). In fact $d\mathbf{u}_r/dr$ is ________ and $d\mathbf{u}_r/d\theta$ is ________.

13 Momentum is $m\mathbf{v}$ and its derivative is $m\mathbf{a}$ = force. Angular momentum is $m\mathbf{H} = m\mathbf{R} \times \mathbf{v}$ and its derivative is ________ = torque. Angular momentum is constant under a central force because the ________ is zero.

14 To find (and remember) $\mathbf{v}$ and $\mathbf{a}$ in polar coordinates, start with the complex number $re^{i\theta}$ and take its derivatives:

$$\mathbf{R} = re^{i\theta} \qquad \frac{d\mathbf{R}}{dt} = \frac{dr}{dt}e^{i\theta} + ir\frac{d\theta}{dt}e^{i\theta}$$

$$\frac{d^2\mathbf{R}}{dt^2} = \_\_\_\_ + \_\_\_\_ + \_\_\_\_ + \_\_\_\_ + \_\_\_\_.$$

**Key idea**: The coefficients of $e^{i\theta}$ and $ie^{i\theta}$ are the $\mathbf{u}_r$ and $\mathbf{u}_\theta$ components of $\mathbf{R}$, $\mathbf{v}$, $\mathbf{a}$:

$$\mathbf{R} = r\mathbf{u}_r + 0\mathbf{u}_\theta \qquad \mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta \qquad \mathbf{a} = \_\_\_\_\_\_\_\_.$$

(a) Fill in the five terms from the derivative of $d\mathbf{R}/dt$

(b) Convert $e^{i\theta}$ to $\mathbf{u}_r$ and $ie^{i\theta}$ to $\mathbf{u}_\theta$ to find $\mathbf{a}$

(c) Compare $\mathbf{R}$, $\mathbf{v}$, $\mathbf{a}$ with formulas (5–6)

(d) (for instructors only) Why does this method work?

Note how $e^{i\theta} = \cos\theta + i\sin\theta$ corresponds to $\mathbf{u}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$. This is one place where electrical engineers are allowed to write $j$ instead of $i$ for $\sqrt{-1}$.

15 If the period is $T$ find from (13) a formula for the distance $a$.

16 To stay above New York what should be the period of a satellite? What should be its distance $a$ from the center of the Earth?

17 From $T$ and $a$ find a formula for the mass $M$.

18 If the moon has a period of 28 days at an average distance of $a = 380{,}000$ km, estimate the mass of the ________.

19 The Earth takes $365\frac{1}{4}$ days to go around the sun at a distance $a \approx 93$ million miles $\approx 150$ million kilometers. Find the mass of the sun.

20 **True or false**:

(a) The paths of all comets are ellipses.

(b) A planet in a circular orbit has constant speed.

(c) Orbits in central force fields are conic sections.

21 $\sqrt{GM} \approx 2 \cdot 10^7$ in what units, based on the Earth's mass $M = 6 \cdot 10^{24}$ kg and the constant $G = 6.67 \cdot 10^{-11}$ Nm$^2$/kg$^2$? A force of one kg · meter/sec$^2$ is a Newton N.

22 If a satellite circles the Earth at 9000 km from the center, estimate its period $T$ in seconds.

23 The Viking 2 orbiter around Mars had a period of about 10,000 seconds. If the mass of Mars is $M = 6.4 \cdot 10^{23}$ kg, what was the value of $a$?

24 Convert $1/r = C - D\cos\theta$, or $1 = Cr - Dx$, into the $xy$ equation of an ellipse.

25 The distances $a$ and $c$ on the ellipse give the constants in $r = 1/(C - D\cos\theta)$. Substitute $\theta = 0$ and $\theta = \pi$ as in Figure 12.10 to find $D = c/(a^2 - c^2)$ and $C = a/(a^2 - c^2) = a/b^2$.

26 Show that $x = -c$, $y = b^2/a$ lies on the ellipse $x^2/a^2 + y^2/b^2 = 1$. Thus $y$ is the height $1/C$ above the sun in Figure 12.10. The distance from the sun to the center has $c^2 = a^2 - b^2$.

27 The point $x = a\cos 2\pi t/T$, $y = b\sin 2\pi t/T$ travels around an ellipse centered at $(0, 0)$ and returns at time $T$. By symmetry it sweeps out area at the same rate at both ends of the major axis. Why does this *break* Kepler's second law?

28 If a central force is $\mathbf{F} = -ma(r)\mathbf{u}_r$, explain why $d^2r/dt^2 - r(d\theta/dt)^2 = -a(r)$. What is $a(r)$ for gravity? Equation (12) for $q = 1/r$ leads to $q_{\theta\theta} + q = r^2 a(r)$.

29 When $F = 0$ the body should travel in a straight ________.

The equation $q_{\theta\theta} + q = 0$ allows $q = \cos\theta$, in which case the path $1/r = \cos\theta$ is ________. Extra credit: Mark off equal distances on a line, connect them to the sun, and explain why the triangles have equal area. So $dA/dt$ is still constant.

**30** The strong nuclear force increases with distance, $a(r) = r$. It binds quarks so tightly that up to now no top quarks have been seen (reliably). Problem 28 gives $q_{\theta\theta} + q = 1/q^3$.

(a) Multiply by $q_\theta$ and integrate to find $\frac{1}{2}q_\theta^2 - \frac{1}{2}q^2 =$ ________ $+ C$.

*(b) Integrate again (with tables) after setting $u = q^2$, $u_\theta = 2qq_\theta$.

**31** The path of a quark in 30(b) can be written as $r^2(A + B\cos 2\theta) = 1$. Show that this is the same as the ellipse $(A+B)x^2 + (A-B)y^2 = 1$ with the origin at the *center.* The nucleus is not at a focus, and the pound note is correct for Newton watching quarks. (Quantum mechanics not accounted for.)

**32** When will Halley's comet appear again? It disappeared in 1986 and its mean distance to the sun (average of $a + c$ and $a - c$) is $a = 1.6 \cdot 10^9$ kilometers.

**33** You are walking at 2 feet/second toward the center of a merry-go-round that turns once every ten seconds. Starting from $r = 20$, $\theta = 0$ fiud $r(t)$, $\theta(t)$, $\mathbf{v}(t)$, $\mathbf{a}(t)$ and the length of your path to the center.

**34** From Kepler's laws $r = 1/(C - D\cos\theta)$ and $r^2 d\theta/dt = h$, show that

**1.** $dr/dt = -Dh\sin\theta$ **2.** $d^2r/dt^2 = \left(\dfrac{1}{r} - C\right)h^2/r^2$

**3.** $d^2r/dt^2 - r(d\theta/dt)^2 = -Ch^2/r^2$.

When Newton reached **3**, he knew that Kepler's laws required a central force of $Ch^2/r^2$. This is his *inverse square law.* Then he went backwards, in our equations (8–12), to show that this force yields Kepler's laws.

**35** How long is our year? The Earth's orbit has $a = 149.57 \cdot 10^6$ kilometers.