

## CHAPTER 14

# Multiple Integrals

### 14.1 Double Integrals

This chapter shows how to integrate functions of two or more variables. First, a double integral is defined as the limit of sums. Second, we find a fast way to compute it. *The key idea is to replace a double integral by two ordinary “single” integrals.*

The double integral  $\iint f(x, y) dy dx$  starts with  $\int f(x, y) dy$ . For each fixed  $x$  we integrate with respect to  $y$ . The answer depends on  $x$ . Now integrate again, this time with respect to  $x$ . The limits of integration need care and attention! Frequently those limits on  $y$  and  $x$  are the hardest part.

Why bother with sums and limits in the first place? Two reasons. There has to be a definition and a computation to fall back on, when the single integrals are difficult or impossible. And also—this we emphasize—*multiple integrals represent more than area and volume*. Those words and the pictures that go with them are the easiest to understand. You can almost see the volume as a “sum of slices” or a “double sum of thin sticks.” The true applications are mostly to other things, but the central idea is always the same: *Add up small pieces and take limits.*

We begin with the area of  $R$  and the volume of  $V$ , by double integrals.

#### A LIMIT OF SUMS

The graph of  $z = f(x, y)$  is a curved surface above the  $xy$  plane. At the point  $(x, y)$  in the plane, the height of the surface is  $z$ . (The surface is *above* the  $xy$  plane only when  $z$  is positive. Volumes below the plane come with minus signs, like areas below the  $x$  axis.) We begin by choosing a positive function—for example  $z = 1 + x^2 + y^2$ .

The base of our solid is a region  $R$  in the  $xy$  plane. That region will be chopped into small rectangles (sides  $\Delta x$  and  $\Delta y$ ). When  $R$  itself is the rectangle  $0 \leq x \leq 1$ ,  $0 \leq y \leq 2$ , the small pieces fit perfectly. For a triangle or a circle, the rectangles miss part of  $R$ . But they do fit in the limit, and any region with a piecewise smooth boundary will be acceptable.

**Question** What is the volume above  $R$  and below the graph of  $z = f(x, y)$ ?

**Answer** It is a double integral—the *integral of  $f(x, y)$  over  $R$* . To reach it we begin with a sum, as suggested by Figure 14.1.

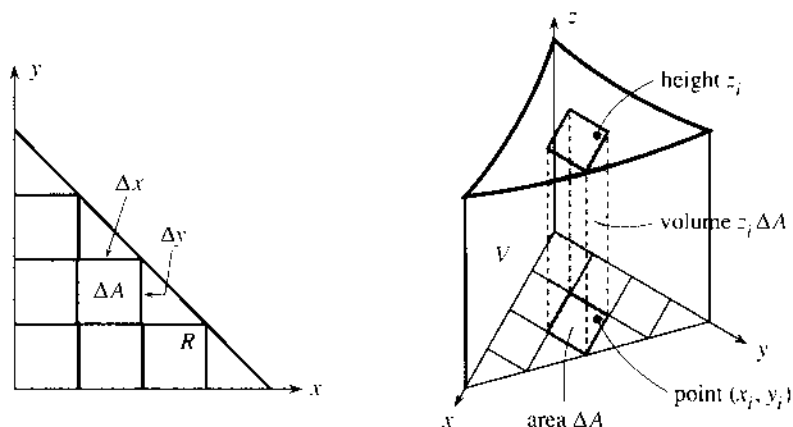


Fig. 14.1 Base  $R$  cut into small pieces  $\Delta A$ . Solid  $V$  cut into thin sticks  $\Delta V = z \Delta A$ .

For single integrals, the interval  $[a, b]$  is divided into short pieces of length  $\Delta x$ . For double integrals,  $R$  is divided into small rectangles of area  $\Delta A = (\Delta x)(\Delta y)$ . Above the  $i$ th rectangle is a “thin stick” with small volume. That volume is the base area  $\Delta A$  times the height above it—except that this height  $z = f(x, y)$  varies from point to point. Therefore we select a point  $(x_i, y_i)$  in the  $i$ th rectangle, and compute the volume from the height above that point:

$$\text{volume of one stick} = f(x_i, y_i)\Delta A \quad \text{volume of all sticks} = \sum f(x_i, y_i)\Delta A.$$

This is the crucial step for any integral—to see it as a sum of small pieces.

Now take limits:  $\Delta x \rightarrow 0$  and  $\Delta y \rightarrow 0$ . The height  $z = f(x, y)$  is nearly constant over each rectangle. (We assume that  $f$  is a continuous function.) The sum approaches a limit, which depends only on the base  $R$  and the surface above it. The limit is the volume of the solid, and it is the **double integral** of  $f(x, y)$  over  $R$ :

$$\iint_R f(x, y) dA = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \sum f(x_i, y_i)\Delta A. \quad (1)$$

To repeat: The limit is the same for all choices of the rectangles and the points  $(x_i, y_i)$ . The rectangles will not fit exactly into  $R$ , if that base area is curved. The heights are not exact, if the surface  $z = f(x, y)$  is also curved. But the errors on the sides and top, where the pieces don’t fit and the heights are wrong, approach zero. Those errors are the volume of the “icing” around the solid, which gets thinner as  $\Delta x \rightarrow 0$  and  $\Delta y \rightarrow 0$ . A careful proof takes more space than we are willing to give. But the properties of the integral need and deserve attention:

1. Linearity:  $\iint (f + g) dA = \iint f dA + \iint g dA$
2. Constant comes outside:  $\iint c f(x, y) dA = c \iint f(x, y) dA$
3.  $R$  splits into  $S$  and  $T$  (not overlapping):  $\iint_R f dA = \iint_S f dA + \iint_T f dA$ .

In 1 the volume under  $f + g$  has two parts. The “thin sticks” of height  $f + g$  split into thin sticks under  $f$  and under  $g$ . In 2 the whole volume is stretched upward by  $c$ . In 3 the volumes are side by side. As with single integrals, these properties help in computations.

By writing  $dA$ , we allow shapes other than rectangles. Polar coordinates have an extra factor  $r$  in  $dA = r dr d\theta$ . By writing  $dx dy$ , we choose rectangular coordinates and prepare for the splitting that comes now.

## SPLITTING A DOUBLE INTEGRAL INTO TWO SINGLE INTEGRALS

The double integral  $\iint f(x, y) dy dx$  will now be reduced to single integrals in  $y$  and then  $x$ . (Or vice versa. Our first integral could equally well be  $\int f(x, y) dx$ .) Chapter 8 described the same idea for solids of revolution. First came the area of a slice, which is a single integral. Then came a second integral to add up the slices. For solids formed by revolving a curve, all slices are circular disks—now we expect other shapes.

Figure 14.2 shows a slice of area  $A(x)$ . It cuts through the solid at a fixed value of  $x$ . The cut starts at  $y = c$  on one side of  $R$ , and ends at  $y = d$  on the other side. This particular example goes from  $y = 0$  to  $y = 2$  ( $R$  is a rectangle). The area of a slice is the  $y$  integral of  $f(x, y)$ . Remember that  $x$  is fixed and  $y$  goes from  $c$  to  $d$ :

$$A(x) = \text{area of slice} = \int_c^d f(x, y) dy \quad (\text{the answer is a function of } x).$$

**EXAMPLE 1**  $A = \int_{y=0}^2 (1 + x^2 + y^2) dy = \left[ y + x^2 y + \frac{y^3}{3} \right]_{y=0}^{y=2} = 2 + 2x^2 + \frac{8}{3}.$

This is the reverse of a partial derivative! The integral of  $x^2 dy$ , with  $x$  constant, is  $x^2 y$ . This “partial integral” is actually called an **inner integral**. After substituting the limits  $y = 2$  and  $y = 0$  and subtracting, we have the area  $A(x) = 2 + 2x^2 + \frac{8}{3}$ . Now the **outer integral** adds slices to find the volume  $\int A(x) dx$ . The answer is a number:

$$\text{volume} = \int_{x=0}^1 \left( 2 + 2x^2 + \frac{8}{3} \right) dx = \left[ 2x + \frac{2x^3}{3} + \frac{8}{3}x \right]_0^1 = 2 + \frac{2}{3} + \frac{8}{3} = \frac{16}{3}.$$

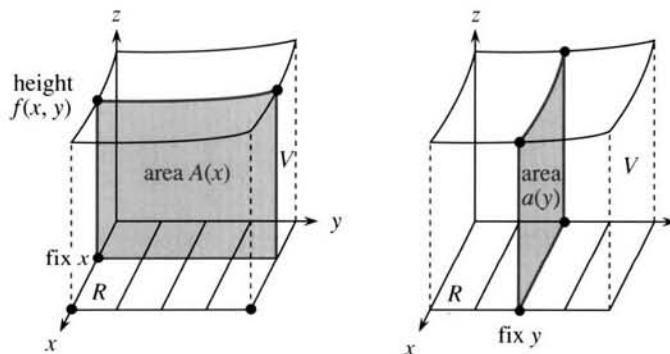


Fig. 14.2 A slice of  $V$  at a fixed  $x$  has area  $A(x) = \int f(x, y) dy$ .

To complete this example, check the volume when the  $x$  integral comes first:

$$\text{inner integral} = \int_{x=0}^1 (1 + x^2 + y^2) dx = \left[ x + \frac{1}{3}x^3 + y^2 x \right]_{x=0}^{x=1} = \frac{4}{3} + y^2$$

$$\text{outer integral} = \int_{y=0}^2 \left( \frac{4}{3} + y^2 \right) dy = \left[ \frac{4}{3}y + \frac{1}{3}y^3 \right]_{y=0}^{y=2} = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}.$$

The fact that double integrals can be split into single integrals is *Fubini's Theorem*.

**14A** If  $f(x, y)$  is continuous on the rectangle  $R$ , then

$$\iint_R f(x, y) dA = \int_a^b \left[ \int_c^d f(x, y) dy \right] dx = \int_c^d \left[ \int_a^b f(x, y) dx \right] dy. \quad (2)$$

The inner integrals are the cross-sectional areas  $A(x)$  and  $a(y)$  of the slices. The outer integrals add up the volumes  $A(x)dx$  and  $a(y)dy$ . Notice the reversing of limits.

Normally the brackets in (2) are omitted. When the  $y$  integral is first,  $dy$  is written inside  $dx$ . **The limits on  $y$  are inside too.** I strongly recommend that you compute the inner integral on one line and the outer integral on a *separate line*.

**EXAMPLE 2** Find the volume below the plane  $z = x - 2y$  and above the base triangle  $R$ .

The triangle  $R$  has sides on the  $x$  and  $y$  axes and the line  $x + y = 1$ . The strips in the  $y$  direction have varying lengths. (So do the strips in the  $x$  direction.) This is the main point of the example—the base is not a rectangle. The upper limit on the inner integral changes as  $x$  changes. **The top of the triangle is at  $y = 1 - x$ .**

Figure 14.3 shows the strips. The region should always be drawn (except for rectangles). Without a figure the limits are hard to find. A sketch of  $R$  makes it easy:

$y$  goes from  $c = 0$  to  $d = 1 - x$ . Then  $x$  goes from  $a = 0$  to  $b = 1$ .

The inner integral has *variable limits* and the outer integral has *constant limits*:

$$\text{inner: } \int_{y=0}^{y=1-x} (x - 2y) dy = \left[ xy - y^2 \right]_{y=0}^{y=1-x} = x(1-x) - (1-x)^2 = -1 + 3x - 2x^2$$

$$\text{outer: } \int_{x=0}^1 (-1 + 3x - 2x^2) dx = \left[ -x + \frac{3}{2}x^2 - \frac{2}{3}x^3 \right]_0^1 = -1 + \frac{3}{2} - \frac{2}{3} = -\frac{1}{6}.$$

The volume is negative. Most of the solid is below the  $xy$  plane. To check the answer  $-\frac{1}{6}$ , do the  $x$  integral first:  $x$  goes from 0 to  $1 - y$ . Then  $y$  goes from 0 to 1.

$$\text{inner: } \int_{x=0}^{1-y} (x - 2y) dx = \left[ \frac{1}{2}x^2 - 2xy \right]_0^{1-y} = \frac{1}{2}(1-y)^2 - 2(1-y)y = \frac{1}{2} - 3y + \frac{5}{2}y^2$$

$$\text{outer: } \int_{y=0}^1 \left( \frac{1}{2} - 3y + \frac{5}{2}y^2 \right) dy = \left[ \frac{1}{2}y - \frac{3}{2}y^2 + \frac{5}{6}y^3 \right]_0^1 = \frac{1}{2} - \frac{3}{2} + \frac{5}{6} = -\frac{1}{6}.$$

Same answer, very probably right. The next example computes  $\iint 1 \, dx \, dy = \text{area of } R$ .

**EXAMPLE 3** The area of  $R$  is  $\int_{x=0}^1 \int_{y=0}^{1-x} dy \, dx$  and also  $\int_{y=0}^1 \int_{x=0}^{1-y} dx \, dy$ .

The first has vertical strips. The inner integral equals  $1 - x$ . Then the outer integral (of  $1 - x$ ) has limits 0 and 1, and the area is  $\frac{1}{2}$ . It is like an indefinite integral inside a definite integral.

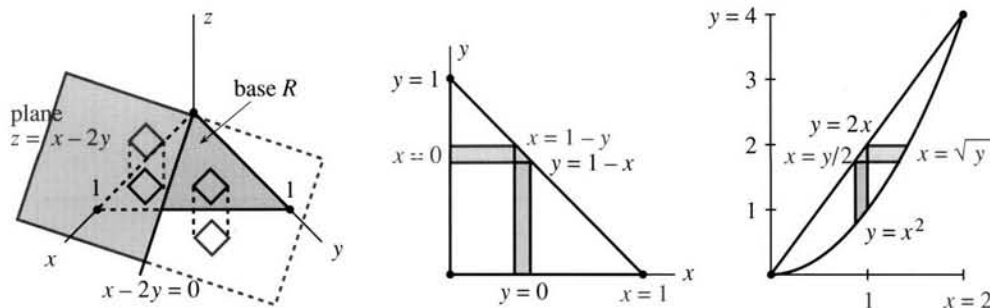


Fig. 14.3 Thin sticks above and below (Example 2). Reversed order (Examples 3 and 4).



**EXAMPLE 4** Reverse the order of integration in  $\int_{x=0}^2 \int_{y=x^2}^{2x} x^3 dy dx$ .

**Solution** Draw a figure! The inner integral goes from the parabola  $y = x^2$  up to the straight line  $y = 2x$ . This gives vertical strips. The strips sit side by side between  $x = 0$  and  $x = 2$ . They stop where  $2x$  equals  $x^2$ , and the line meets the parabola.

The problem is to put the  $x$  integral first. It goes along horizontal strips. On each line  $y = \text{constant}$ , we need the *entry value* of  $x$  and the *exit value* of  $x$ . From the figure,  $x$  goes from  $\frac{1}{2}y$  to  $\sqrt{y}$ . Those are the inner limits. Pay attention also to the outer limits, because they now apply to  $y$ . The region starts at  $y = 0$  and ends at  $y = 4$ . *No change in the integrand  $x^3$ —that is the height of the solid:*

$$\int_{x=0}^2 \int_{y=x^2}^{2x} x^3 dy dx \text{ is reversed to } \int_{y=0}^4 \int_{x=\frac{1}{2}y}^{\sqrt{y}} x^3 dx dy. \quad (3)$$

**EXAMPLE 5** Find the volume bounded by the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$ , and  $2x + y + z = 4$ .

**Solution** The solid is a tetrahedron (four sides). It goes from  $z = 0$  (the  $xy$  plane) up to the plane  $2x + y + z = 4$ . On that plane  $z = 4 - 2x - y$ . This is the height function  $f(x, y)$  to be integrated.

Figure 14.4 shows the base  $R$ . To find its sides, set  $z = 0$ . The sides of  $R$  are the lines  $x = 0$  and  $y = 0$  and  $2x + y = 4$ . Taking vertical strips,  $dy$  is inner:

$$\begin{aligned} \text{inner: } \int_{y=0}^{4-2x} (4-2x-y) dy &= \left[ (4-2x)y - \frac{1}{2}y^2 \right]_0^{4-2x} = \frac{1}{2}(4-2x)^2 \\ \text{outer: } \int_{x=0}^2 \frac{1}{2}(4-2x)^2 dx &= \left[ -\frac{(4-2x)^3}{2 \cdot 3 \cdot 2} \right]_0^2 = \frac{4^3}{2 \cdot 3 \cdot 2} = \frac{16}{3}. \end{aligned}$$

**Question** What is the meaning of the inner integral  $\frac{1}{2}(4-2x)^2$  (and also  $\frac{16}{3}$ )?

**Answer** The first is  $A(x)$ , the area of the slice.  $\frac{16}{3}$  is the solid volume.

**Question** What if the inner integral  $\int f(x, y) dy$  has limits that depend on  $y$ ?

**Answer** It can't. Those limits must be wrong. Find them again.

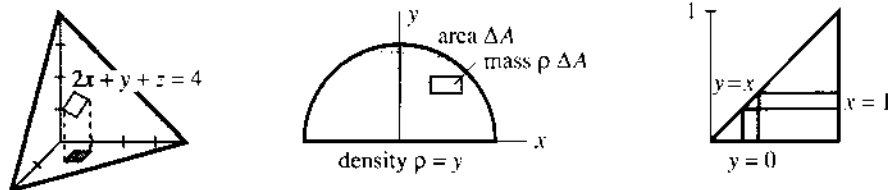


Fig. 14.4 Tetrahedron in Example 5, semicircle in Example 6, triangle in Example 7.

**EXAMPLE 6** Find the mass in a semicircle  $0 \leq y \leq \sqrt{1-x^2}$  if the density is  $\rho = y$ .

This is a new application of double integrals. The total mass is a sum of small masses ( $\rho$  times  $\Delta A$ ) in rectangles of area  $\Delta A$ . The rectangles don't fit perfectly inside the semicircle  $R$ , and the density is not constant in each rectangle—but those problems

disappear in the limit. We are left with a double integral:

$$\text{total mass } M = \iint_R \rho \, dA = \iint_R \rho(x, y) \, dx \, dy. \quad (4)$$

Set  $\rho = y$ . Figure 14.4 shows the limits on  $x$  and  $y$  (try both  $dy \, dx$  and  $dx \, dy$ ):

$$\text{mass } M = \int_{x=-1}^1 \int_{y=0}^{\sqrt{1-x^2}} y \, dy \, dx \quad \text{and also} \quad M = \int_{y=0}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y \, dx \, dy.$$

The first inner integral is  $\frac{1}{2}y^2$ . Substituting the limits gives  $\frac{1}{2}(1-x^2)$ . The outer integral of  $\frac{1}{2}(1-x^2)$  yields the total mass  $M = \frac{2}{3}$ .

The second inner integral is  $xy$ . Substituting the limits on  $x$  gives \_\_\_\_\_. Then the outer integral is  $-\frac{2}{3}(1-y^2)^{3/2}$ . Substituting  $y = 1$  and  $y = 0$  yields  $M = \frac{2}{3}$ .

**Remark** This same calculation also produces the *moment* around the  $x$  axis, when the density is  $\rho = 1$ . The factor  $y$  is the distance to the  $x$  axis. **The moment is**  $M_x = \iint y \, dA = \frac{2}{3}$ . Dividing by the area of the semicircle (which is  $\pi/2$ ) locates the centroid:  $\bar{x} = 0$  by symmetry and

$$\bar{y} = \text{height of centroid} = \frac{\text{moment}}{\text{area}} = \frac{2/3}{\pi/2} = \frac{4}{3\pi}. \quad (5)$$

This is the “average height” of points inside the semicircle, found earlier in 8.5.

**EXAMPLE 7** Integrate  $\int_{y=0}^{y=1} \int_{x=y}^{x=1} \cos x^2 \, dx \, dy$  avoiding the impossible  $\int \cos x^2 \, dx$ .

This is a famous example where reversing the order makes the calculation possible. The base  $R$  is the triangle in Figure 14.4 (note that  $x$  goes from  $y$  to 1). **In the opposite order**  $y$  goes from 0 to  $x$ . Then  $\int \cos x^2 \, dy = y \cos x^2$  contains the factor  $x$  that we need:

$$\text{outer integral: } \int_0^1 x \cos x^2 \, dx = \left[ \frac{1}{2} \sin x^2 \right]_0^1 = \frac{1}{2} \sin 1.$$

## 14.1 EXERCISES

### Read-through questions

The double integral  $\iint_R f(x, y) \, dA$  gives the volume between  $R$  and \_\_\_\_\_. The base is first cut into small \_\_\_\_\_ of area  $\Delta A$ . The volume above the  $i$ th piece is approximately \_\_\_\_\_. The limit of the sum \_\_\_\_\_ is the volume integral. Three properties of double integrals are \_\_\_\_\_ (linearity) and \_\_\_\_\_ and \_\_\_\_\_.

If  $R$  is the rectangle  $0 \leq x \leq 4$ ,  $4 \leq y \leq 6$ , the integral  $\iint x \, dA$  can be computed two ways. One is  $\iint x \, dy \, dx$ , when the inner integral is  $\int_4^6 x \, dy = \underline{\hspace{1cm}}$ . The outer integral gives  $\int_0^4 \underline{\hspace{1cm}} \, dx = \underline{\hspace{1cm}}$ . When the  $x$  integral comes first it equals  $\int_0^4 x \, dx = \underline{\hspace{1cm}}$ . Then the  $y$  integral equals  $\int_4^6 \underline{\hspace{1cm}} \, dy = \underline{\hspace{1cm}}$ . This is the volume between \_\_\_\_\_ (describe  $V$ ).

The area of  $R$  is  $\iint \underline{\hspace{1cm}} \, dy \, dx$ . When  $R$  is the triangle between  $x = 0$ ,  $y = 2x$ , and  $y = 1$ , the inner limits on  $y$  are \_\_\_\_\_. This is the length of a \_\_\_\_\_ strip. The (outer) limits on  $x$  are \_\_\_\_\_. The area is \_\_\_\_\_. In the opposite order, the

(inner) limits on  $x$  are \_\_\_\_\_. Now the strip is \_\_\_\_\_ and the outer integral is \_\_\_\_\_. When the density is  $\rho(x, y)$ , the total mass in the region  $R$  is  $\iint \underline{\hspace{1cm}} \, dA$ . The moments are  $M_y = \underline{\hspace{1cm}}$  and  $M_x = \underline{\hspace{1cm}}$ . The centroid has  $\bar{x} = M_y/M$ .

Compute the double integrals 1–4 by two integrations.

1  $\int_{y=0}^1 \int_{x=0}^2 x^2 \, dx \, dy$  and  $\int_{y=0}^1 \int_{x=0}^2 y^2 \, dx \, dy$

2  $\int_{y=2}^{2e} \int_{x=1}^e 2xy \, dx \, dy$  and  $\int_{y=2}^{2e} \int_{x=1}^e dx \, dy / xy$

3  $\int_0^{\pi/2} \int_0^{\pi/4} \sin(x+y) \, dx \, dy$  and  $\int_1^2 \int_0^2 dy \, dx / (x+y)^2$

4  $\int_0^1 \int_1^2 ye^{xy} \, dx \, dy$  and  $\int_{-1}^1 \int_0^3 dy \, dx / \sqrt{3+2x+y}$

In 5–10, draw the region and compute the area.

$$5 \int_{x=1}^2 \int_{y=1}^{2x} dy \, dx$$

$$6 \int_0^1 \int_{x^3}^x dy \, dx$$

$$7 \int_0^\infty \int_{e^{-2x}}^{e^{-x}} dy \, dx$$

$$8 \int_{-1}^1 \int_{x^2-1}^{1-x^2} dy \, dx$$

$$9 \int_{-1}^1 \int_{y^2}^1 dx \, dy$$

$$10 \int_{-1}^1 \int_{x=y}^{|y|} dx \, dy$$

In 11–16 reverse the order of integration (and find the new limits) in 5–10 respectively.

In 17–24 find the limits on  $\iint dy \, dx$  and  $\iint dx \, dy$ . Draw  $R$  and compute its area.

17  $R$  = triangle inside the lines  $x=0$ ,  $y=1$ ,  $y=2x$ .

18  $R$  = triangle inside the lines  $x=-1$ ,  $y=0$ ,  $x+y=0$ .

19  $R$  = triangle inside the lines  $y=x$ ,  $y=-x$ ,  $y=3$ .

20  $R$  = triangle inside the lines  $y=x$ ,  $y=2x$ ,  $y=4$ .

21  $R$  = triangle with vertices  $(0, 0)$ ,  $(4, 4)$ ,  $(4, 8)$ .

22  $R$  = triangle with vertices  $(0, 0)$ ,  $(-2, -1)$ ,  $(1, -2)$ .

23  $R$  = triangle with vertices  $(0, 0)$ ,  $(2, 0)$ ,  $(1, b)$ . Here  $b > 0$ .

\*24  $R$  = triangle with vertices  $(0, 0)$ ,  $(a, b)$ ,  $(c, d)$ . The sides are  $y=bx/a$ ,  $y=dx/c$ , and  $y=b+(x-a)(d-b)/(c-a)$ . Find  $A = \iint dy \, dx$  when  $0 < a < c$ ,  $0 < d < b$ .

$$25 \text{ Evaluate } \int_0^b \int_0^a \partial^2 f / \partial x \partial y \, dx \, dy.$$

$$26 \text{ Evaluate } \int_0^b \int_0^a \partial f / \partial x \, dx \, dy.$$

In 27–28, divide the unit square  $R$  into triangles  $S$  and  $T$  and verify  $\iint_R f \, dA = \iint_S f \, dA + \iint_T f \, dA$ .

$$27 \, f(x, y) = 2x - 3y + 1 \quad 28 \, f(x, y) = xe^y - ye^x$$

29 The area under  $y=f(x)$  is a single integral from  $a$  to  $b$  or a double integral (find the limits):

$$\int_a^b f(x) \, dx = \iint 1 \, dy \, dx.$$

30 Find the limits and the area under  $y=1-x^2$ :

$$\int (1-x^2) \, dx \text{ and } \iint 1 \, dx \, dy \text{ (reversed from 29).}$$

31 A city inside the circle  $x^2 + y^2 = 100$  has population density  $\rho(x, y) = 10(100 - x^2 - y^2)$ . Integrate to find its population.

32 Find the volume bounded by the planes  $x=0$ ,  $y=0$ ,  $z=0$ , and  $ax+by+cz=1$ .

In 33–34 the rectangle with corners  $(1, 1)$ ,  $(1, 3)$ ,  $(2, 1)$ ,  $(2, 3)$  has density  $\rho(x, y) = x^2$ . The moments are  $M_y = \iint x \rho \, dA$  and  $M_x = \iint y \rho \, dA$ .

33 Find the mass.

34 Find the center of mass.

In 35–36 the region is a circular wedge of radius 1 between the lines  $y=x$  and  $y=-x$ .

35 Find the area.

36 Find the centroid  $(\bar{x}, \bar{y})$ .

37 Write a program to compute  $\int_0^1 \int_0^1 f(x, y) \, dx \, dy$  by the midpoint rule (midpoints of  $n^2$  small squares). Which  $f(x, y)$  are integrated exactly by your program?

38 Apply the midpoint code to integrate  $x^2$  and  $xy$  and  $y^2$ . The errors decrease like what power of  $\Delta x = \Delta y = 1/n$ ?

Use the program to compute the volume under  $f(x, y)$  in 39–42. Check by integrating exactly or doubling  $n$ .

$$39 \, f(x, y) = 3x + 4y + 5$$

$$40 \, f(x, y) = 1/\sqrt{x^2 + y^2}$$

$$41 \, f(x, y) = x^y$$

$$42 \, f(x, y) = e^x \sin \pi y$$

43 In which order is  $\iint x^y \, dx \, dy = \iint x^y \, dy \, dx$  easier to integrate over the square  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ ? By reversing order, integrate  $(x-1)/\ln x$  from 0 to 1—its antiderivative is unknown.

44 Explain in your own words the definition of the double integral of  $f(x, y)$  over the region  $R$ .

45  $\sum y_i \Delta A$  might not approach  $\iint y \, dA$  if we only know that  $\Delta A \rightarrow 0$ . In the square  $0 \leq x, y \leq 1$ , take rectangles of sides  $\Delta x$  and 1 (not  $\Delta x$  and  $\Delta y$ ). If  $(x_i, y_i)$  is a point in the rectangle where  $y_i = 1$ , then  $\sum y_i \Delta A =$  \_\_\_\_\_. But  $\iint y \, dA =$  \_\_\_\_\_.

## 14.2 Change to Better Coordinates

You don't go far with double integrals before wanting to *change variables*. Many regions simply do not fit with the  $x$  and  $y$  axes. Two examples are in Figure 14.5, a tilted square and a ring. Those are excellent shapes—in the *right coordinates*.

We have to be able to answer basic questions like these:

Find the area  $\iint dA$  and moment  $\iint x dA$  and moment of inertia  $\iint x^2 dA$ .

The problem is: *What is  $dA$ ?* We are leaving the  $xy$  variables where  $dA = dx dy$ .

The reason for changing is this: The limits of integration in the  $y$  direction are miserable. I don't know them and I don't want to know them. For every  $x$  we would need the entry point  $P$  of the line  $x = \text{constant}$ , and the exit point  $Q$ . The heights of  $P$  and  $Q$  are the limits on  $\int dy$ , the inner integral. The geometry of the square and ring are totally missed, if we stick rigidly to  $x$  and  $y$ .

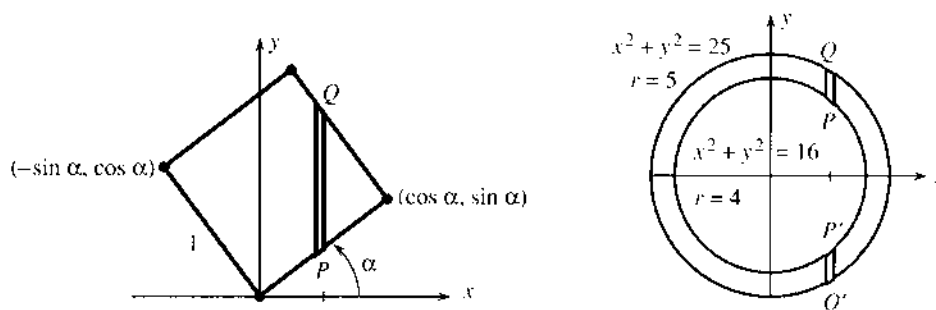


Fig. 14.5 Unit square turned through angle  $\alpha$ . Ring with radii 4 and 5.

Which coordinates are better? Any sensible person agrees that the area of the tilted square is 1. "Just turn it and the area is obvious." But that sensible person may not know the moment or the center of gravity or the moment of inertia. So we actually have to do the turning.

The new coordinates  $u$  and  $v$  are in Figure 14.6a. The limits of integration on  $v$  are 0 and 1. So are the limits on  $u$ . *But when you change variables, you don't just change limits.* Two other changes come with new variables:

1. The small area  $dA = dx dy$  becomes  $dA = \underline{\hspace{2cm}} du dv$ .
2. The integral of  $x$  becomes the integral of  $\underline{\hspace{2cm}}$ .

Substituting  $u = \sqrt{x}$  in a single integral, we make the same changes. Limits  $x = 0$  and  $x = 4$  become  $u = 0$  and  $u = 2$ . Since  $x$  is  $u^2$ ,  $dx$  is  $2u du$ . The purpose of the change is to find an antiderivative. For double integrals, the usual purpose is to improve the limits—but we have to accept the whole package.

To turn the square, there are formulas connecting  $x$  and  $y$  to  $u$  and  $v$ . The geometry is clear—**rotate axes by  $\alpha$** —but it has to be converted into algebra:

$$\begin{aligned} u &= x \cos \alpha + y \sin \alpha & \text{and in reverse} & & x &= u \cos \alpha - v \sin \alpha \\ v &= -x \sin \alpha + y \cos \alpha & & & y &= u \sin \alpha + v \cos \alpha. \end{aligned} \quad (1)$$

Figure 14.6 shows the rotation. As points move, the whole square turns. A good way to remember equation (1) is to follow the corners as they become  $(1, 0)$  and  $(0, 1)$ .

The change from  $\iint x dA$  to  $\iint \underline{\hspace{2cm}} du dv$  is partly decided by equation (1). It gives  $x$  as a function of  $u$  and  $v$ . We also need  $dA$ . For a pure rotation the first guess is correct: *The area  $dx dy$  equals the area  $du dv$ .* For most changes of variable this is *false*. The general formula for  $dA$  comes after the examples.

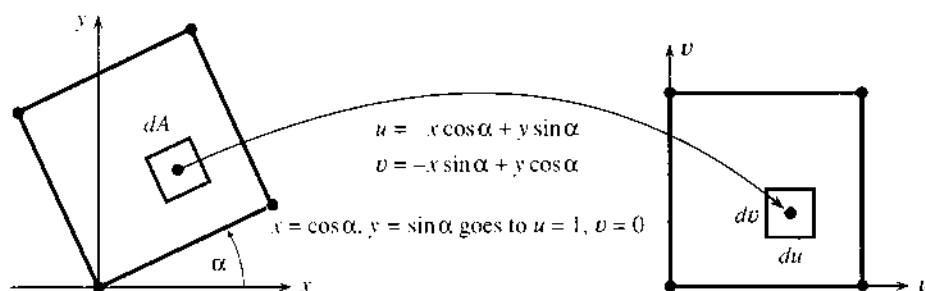


Fig. 14.6 Change of coordinates—axes turned by  $\alpha$ . For rotation  $dA$  is  $du dv$ .

**EXAMPLE 1** Find  $\iint dA$  and  $\iint x dA$  and  $\bar{x}$  and also  $\iint x^2 dA$  for the tilted square.

**Solution** The area of the square is  $\int_0^1 \int_0^1 du dv = 1$ . Notice the good limits. Then

$$\iint x dA = \int_0^1 \int_0^1 (u \cos \alpha - v \sin \alpha) du dv = \frac{1}{2} \cos \alpha - \frac{1}{2} \sin \alpha. \quad (2)$$

This is the *moment around the y axis*. The factors  $\frac{1}{2}$  come from  $\frac{1}{2}u^2$  and  $\frac{1}{2}v^2$ . The  $x$  coordinate of the center of gravity is

$$\bar{x} = \iint x dA / \iint dA = (\frac{1}{2} \cos \alpha - \frac{1}{2} \sin \alpha) / 1.$$

Similarly the integral of  $y$  leads to  $\bar{y}$ . The answer is no mystery—the point  $(\bar{x}, \bar{y})$  is at the center of the square! Substituting  $x = u \cos \alpha - v \sin \alpha$  made  $x dA$  look worse, but the limits 0 and 1 are much better.

The moment of inertia  $I_y$  around the  $y$  axis is also simplified:

$$\iint x^2 dA = \int_0^1 \int_0^1 (u \cos \alpha - v \sin \alpha)^2 du dv = \frac{\cos^2 \alpha}{3} - \frac{\cos \alpha \sin \alpha}{2} + \frac{\sin^2 \alpha}{3}. \quad (3)$$

You know this next fact but I will write it anyway: *The answers don't contain  $u$  or  $v$ .* Those are dummy variables like  $x$  and  $y$ . The answers do contain  $\alpha$ , because the square has turned. (The area is fixed at 1.) The moment of inertia  $I_x = \iint y^2 dA$  is the same as equation (3) but with all plus signs.

**Question** The sum  $I_x + I_y$  simplifies to  $\frac{2}{3}$  (a constant). Why no dependence on  $\alpha$ ?

**Answer**  $I_x + I_y$  equals  $I_0$ . This moment of inertia around  $(0, 0)$  is unchanged by rotation. We are turning the square around one of its corners.

## CHANGE TO POLAR COORDINATES

**The next change is to  $r$  and  $\theta$ .** A small area becomes  $dA = r dr d\theta$  (definitely not  $dr d\theta$ ). Area always comes from multiplying two lengths, and  $d\theta$  is not a length. Figure 14.7 shows the crucial region—a “polar rectangle” cut out by rays and circles. Its area  $\Delta A$  is found in two ways, both leading to  $r dr d\theta$ :

(*Approximate*) The straight sides have length  $\Delta r$ . The circular arcs are close to  $r\Delta\theta$ . The angles are  $90^\circ$ . So  $\Delta A$  is close to  $(\Delta r)(r\Delta\theta)$ .

(*Exact*) A wedge has area  $\frac{1}{2}r^2\Delta\theta$ . The difference between wedges is  $\Delta A$ :

$$\Delta A = \frac{1}{2} \left( r + \frac{\Delta r}{2} \right)^2 \Delta\theta - \frac{1}{2} \left( r - \frac{\Delta r}{2} \right)^2 \Delta\theta = r \Delta r \Delta\theta.$$

The exact method places  $r$  dead center (see figure). The approximation says: Forget the change in  $r\Delta\theta$  as you move outward. Keep only the first-order terms.

A third method is coming, which requires no picture and no geometry. Calculus always has a third method! The change of variables  $x = r \cos \theta$ ,  $y = r \sin \theta$  will go into a general formula for  $dA$ , and out will come the area  $r dr d\theta$ .

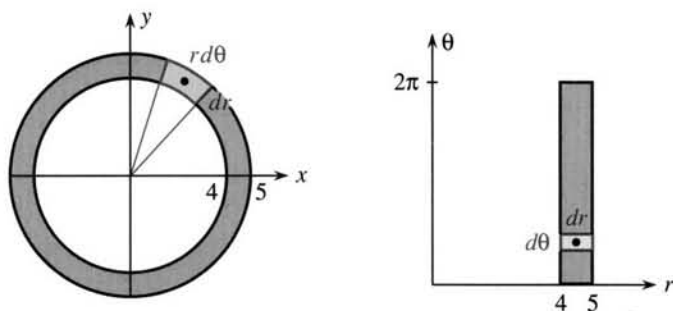


Fig. 14.7 Ring and polar rectangle in  $xy$  and  $r\theta$ , with stretching factor  $r = 4.5$ .

**EXAMPLE 2** Find the area and center of gravity of the ring. Also find  $\iint x^2 dA$ .

**Solution** The limits on  $r$  are 4 and 5. The limits on  $\theta$  are 0 and  $2\pi$ . *Polar coordinates are perfect for a ring.* Compared with limits like  $x = \sqrt{25 - y^2}$ , the change to  $r dr d\theta$  is a small price to pay:

$$\text{area} = \int_0^{2\pi} \int_4^5 r dr d\theta = 2\pi \left[ \frac{1}{2} r^2 \right]_4^5 = \pi 5^2 - \pi 4^2 = 9\pi.$$

The  $\theta$  integral is  $2\pi$  (full circle). Actually the ring is a giant polar rectangle. We could have used the exact formula  $r \Delta r \Delta \theta$ , with  $\Delta \theta = 2\pi$  and  $\Delta r = 5 - 4$ . When the radius  $r$  is centered at 4.5, the product  $r \Delta r \Delta \theta$  is  $(4.5)(1)(2\pi) = 9\pi$  as above.

Since the ring is symmetric around  $(0, 0)$ , the integral of  $x dA$  must be zero:

$$\iint_R x dA = \int_0^{2\pi} \int_4^5 (r \cos \theta) r dr d\theta = \left[ \frac{1}{3} r^3 \right]_4^5 \left[ \sin \theta \right]_0^{2\pi} = 0.$$

Notice  $r \cos \theta$  from  $x$ —the other  $r$  is from  $dA$ . The moment of inertia is

$$\iint_R x^2 dA = \int_0^{2\pi} \int_4^5 r^2 \cos^2 \theta r dr d\theta = \left[ \frac{1}{4} r^4 \right]_4^5 \int_0^{2\pi} \cos^2 \theta d\theta = \frac{1}{4} (5^4 - 4^4) \pi.$$

This  $\theta$  integral is  $\pi$  not  $2\pi$ , because the average of  $\cos^2 \theta$  is  $\frac{1}{2}$  not 1.

For reference here are the moments of inertia when the density is  $\rho(x, y)$ :

$$I_y = \iint x^2 \rho dA \quad I_x = \iint y^2 \rho dA \quad I_0 = \iint r^2 \rho dA = \text{polar moment} = I_x + I_y. \quad (4)$$

**EXAMPLE 3** Find masses and moments for semicircular plates:  $\rho = 1$  and  $\rho = 1 - r$ .

**Solution** The semicircles in Figure 14.8 have  $r = 1$ . The angle goes from 0 to  $\pi$  (the upper half-circle). Polar coordinates are best. **The mass is the integral of the density  $\rho$ :**

$$M = \int_0^\pi \int_0^1 r dr d\theta = \left( \frac{1}{2} \right) (\pi) \quad \text{and} \quad M = \int_0^\pi \int_0^1 (1 - r) r dr d\theta = \left( \frac{1}{6} \right) (\pi).$$

The first mass  $\pi/2$  equals the area (because  $\rho = 1$ ). The second mass  $\pi/6$  is smaller (because  $\rho < 1$ ). Integrating  $\rho = 1$  is the same as finding a volume when the height is  $z = 1$  (part of a cylinder). Integrating  $\rho = 1 - r$  is the same as finding a volume when the height is  $z = 1 - r$  (part of a cone). Volumes of cones have the extra factor  $\frac{1}{3}$ .

The center of gravity involves the moment  $M_x = \iint y \rho \, dA$ . *The distance from the  $x$  axis is  $y$ , the mass of a small piece is  $\rho \, dA$ , integrate to add mass times distance.* Polar coordinates are still best, with  $y = r \sin \theta$ . Again  $\rho = 1$  and  $\rho = 1 - r$ :

$$\iint y \, dA = \int_0^\pi \int_0^1 r \sin \theta \, r \, dr \, d\theta = \frac{2}{3} \quad \iint y(1-r) \, dA = \int_0^\pi \int_0^1 r \sin \theta (1-r) \, r \, dr \, d\theta = \frac{1}{6}.$$

The height of the center of gravity is  $\bar{y} = M_x/M = \text{moment divided by mass}$ :

$$\bar{y} = \frac{2/3}{\pi/2} = \frac{4}{3\pi} \quad \text{when } \rho = 1 \quad \bar{y} = \frac{1/6}{\pi/6} = \frac{1}{\pi} \quad \text{when } \rho = 1 - r.$$

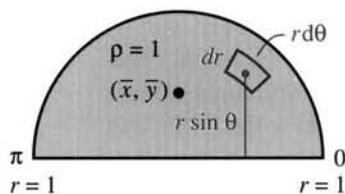


Fig. 14.8 Semicircles with density piled above them.

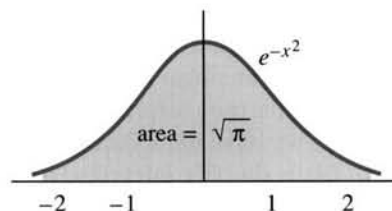
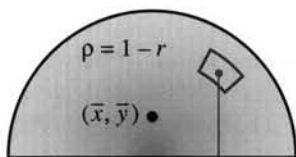


Fig. 14.9 Bell-shaped curve.

**Question** Compare  $\bar{y}$  for  $\rho = 1$  and  $\rho = \text{other positive constants}$  and  $\rho = 1 - r$ .

**Answer** Any constant  $\rho$  gives  $\bar{y} = 4/3\pi$ . Since  $1 - r$  is dense at  $r = 0$ ,  $\bar{y}$  drops to  $1/\pi$ .

**Question** How is  $\bar{y} = 4/3\pi$  related to the “average” of  $y$  in the semicircle?

**Answer** They are identical. This is the point of  $\bar{y}$ . Divide the integral by the area:

$$\text{The average value of a function is } \frac{\iint f(x, y) \, dA}{\iint dA}. \quad (5)$$

The integral of  $f$  is divided by the integral of 1 (the area). In one dimension  $\int_a^b v(x) \, dx$  was divided by  $\int_a^b 1 \, dx$  (the length  $b - a$ ). That gave the average value of  $v(x)$  in Section 5.6. Equation (5) is the same idea for  $f(x, y)$ .

**EXAMPLE 4** Compute  $A = \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$  from  $A^2 = \int_{-\infty}^{\infty} e^{-x^2} \, dx \int_{-\infty}^{\infty} e^{-y^2} \, dy = \pi$ .

$A$  is the area under a “bell-shaped curve”—see Figure 14.9. This is the most important definite integral in the study of probability. It is difficult because a factor  $2x$  is not present. Integrating  $2xe^{-x^2}$  gives  $-e^{-x^2}$ , but integrating  $e^{-x^2}$  is impossible—except approximately by a computer. How can we hope to show that  $A$  is exactly  $\sqrt{\pi}$ ?

The trick is to go from an area integral  $A$  to a volume integral  $A^2$ . This is unusual (and hard to like), but the end justifies the means:

$$A^2 = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{-x^2} e^{-y^2} \, dy \, dx = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2} r \, dr \, d\theta. \quad (6)$$

The double integrals cover the whole plane. The  $r^2$  comes from  $x^2 + y^2$ , and the key factor  $r$  appears in polar coordinates. It is now possible to substitute  $u = r^2$ . The  $r$  integral is  $\frac{1}{2} \int_0^\infty e^{-u} \, du = \frac{1}{2}$ . The  $\theta$  integral is  $2\pi$ . The double integral is  $(\frac{1}{2})(2\pi)$ . Therefore  $A^2 = \pi$  and the single integral is  $A = \sqrt{\pi}$ .

**EXAMPLE 5** Apply Example 4 to the “normal distribution”  $p(x) = e^{-x^2/2}/\sqrt{2\pi}$ .

Section 8.4 discussed probability. It emphasized the importance of this particular  $p(x)$ . At that time we could not verify that  $\int p(x) dx = 1$ . Now we can:

$$x = \sqrt{2}y \quad \text{yields} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy = 1. \quad (7)$$

**Question** Why include the 2's in  $p(x)$ ? The integral of  $e^{-x^2}/\sqrt{\pi}$  also equals 1.

**Answer** With the 2's the “variance” is  $\int x^2 p(x) dx = 1$ . This is a convenient number.

### CHANGE TO OTHER COORDINATES

A third method was promised, to find  $r dr d\theta$  without a picture and without geometry. The method works directly from  $x = r \cos \theta$  and  $y = r \sin \theta$ . It also finds the 1 in  $du dv$ , after a rotation of axes. Most important, this new method finds the factor  $J$  in the area  $dA = J du dv$ , for any change of variables. The change is from  $xy$  to  $uv$ .

For single integrals, the “stretching factor”  $J$  between the original  $dx$  and the new  $du$  is (not surprisingly) the ratio  $dx/du$ . Where we have  $dx$ , we write  $(dx/du)du$ . Where we have  $(du/dx)dx$ , we write  $du$ . That was the idea of substitutions—the main way to simplify integrals.

For double integrals the stretching factor appears in the area:  $dx dy$  becomes  $|J| du dv$ . *The old and new variables are related by  $x = x(u, v)$  and  $y = y(u, v)$ .* The point with coordinates  $u$  and  $v$  comes from the point with coordinates  $x$  and  $y$ . A whole region  $S$ , full of points in the  $uv$  plane, comes from the region  $R$  full of corresponding points in the  $xy$  plane. A small piece with area  $|J| du dv$  comes from a small piece with area  $dx dy$ . *The formula for  $J$  is a two-dimensional version of  $dx/du$ .*

**14B** The stretching factor for area is the 2 by 2 **Jacobian determinant**  $J(u, v)$ :

$$J = \begin{vmatrix} \partial x / \partial u & \partial x / \partial v \\ \partial y / \partial u & \partial y / \partial v \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}. \quad (8)$$

An integral over  $R$  in the  $xy$  plane becomes an integral over  $S$  in the  $uv$  plane:

$$\iint_R f(x, y) dx dy = \iint_S f(x(u, v), y(u, v)) |J| du dv. \quad (9)$$

The determinant  $J$  is often written  $\partial(x, y)/\partial(u, v)$ , as a reminder that this stretching factor is like  $dx/du$ . We require  $J \neq 0$ . That keeps the stretching and shrinking under control.

You naturally ask: Why take the absolute value  $|J|$  in equation (9)? Good question—it wasn't done for single integrals. The reason is in the limits of integration. The single integral  $\int_0^1 dx$  is  $\int_0^{-1} (-du)$  after changing  $x$  to  $-u$ . We keep the minus sign and allow single integrals to run backward. Double integrals could too, but normally they go left to right and down to up. We use the absolute value  $|J|$  and run forward.

**EXAMPLE 6** Polar coordinates have  $x = u \cos v = r \cos \theta$  and  $y = u \sin v = r \sin \theta$ .

With no geometry: 
$$J = \begin{vmatrix} \partial x / \partial r & \partial x / \partial \theta \\ \partial y / \partial r & \partial y / \partial \theta \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r. \quad (10)$$



**EXAMPLE 7** Find  $J$  for the linear change to  $x = au + bv$  and  $y = cu + dv$ .

Ordinary determinant:  $J = \begin{vmatrix} \partial x / \partial u & \partial x / \partial v \\ \partial y / \partial u & \partial y / \partial v \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \quad (11)$

Why make this simple change, in which  $a, b, c, d$  are all constant? It straightens parallelograms into squares (and rotates those squares). Figure 14.10 is typical.

Common sense indicated  $J = 1$  for pure rotation—no change in area. Now  $J = 1$  comes from equations (1) and (11), because  $ad - bc$  is  $\cos^2 \alpha + \sin^2 \alpha$ .

In practice,  $xy$  rectangles generally go into  $uv$  rectangles. The sides can be curved (as in polar rectangles) but the angles are often  $90^\circ$ . The change is “orthogonal.” The next example has angles that are not  $90^\circ$ , and  $J$  still gives the answer.

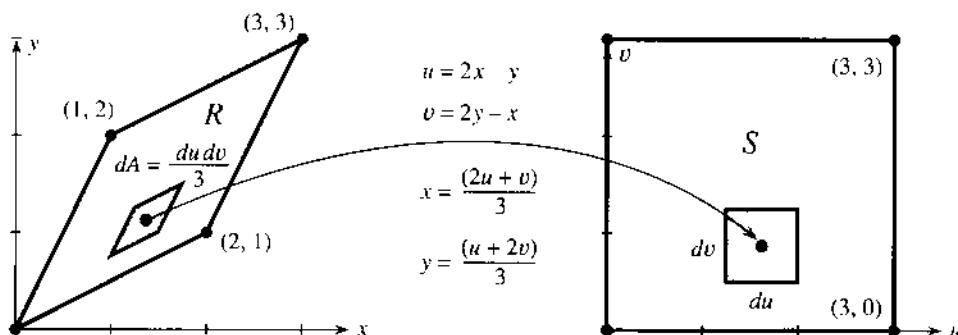


Fig. 14.10 Change from  $xy$  to  $uv$  has  $J = \frac{1}{3}$ .

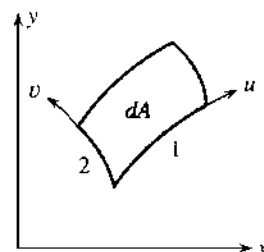


Fig. 14.11 Curved areas are also  $dA = |J| du dv$ .

**EXAMPLE 8** Find the area of  $R$  in Figure 14.10. Also compute  $\iint_R e^x dx dy$ .

**Solution** The figure shows  $x = \frac{2}{3}u + \frac{1}{3}v$  and  $y = \frac{1}{3}u + \frac{2}{3}v$ . The determinant is

$$J = \begin{vmatrix} \partial x / \partial u & \partial x / \partial v \\ \partial y / \partial u & \partial y / \partial v \end{vmatrix} = \begin{vmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{vmatrix} = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}.$$

The area of the  $xy$  parallelogram becomes an integral over the  $uv$  square:

$$\iint_R dx dy = \iint_S |J| du dv = \int_0^3 \int_0^3 \frac{1}{3} du dv = \frac{1}{3} \cdot 3 \cdot 3 = 3.$$

The square has area 9, the parallelogram has area 3. I don't know if  $J = \frac{1}{3}$  is a stretching factor or a shrinking factor. The other integral  $\iint e^x dx dy$  is

$$\int_0^3 \int_0^3 e^{2u/3 + v/3} \frac{1}{3} du dv = \left[ \frac{3}{2} e^{2u/3} \right]_0^3 \left[ 3e^{v/3} \right]_0^3 \frac{1}{3} = \frac{3}{2} (e^2 - 1)(e - 1).$$

**Main point:** The change to  $u$  and  $v$  makes the limits easy (just 0 and 3).

**Why is the stretching factor  $J$  a determinant?** With straight sides, this goes back to Section 11.3 on vectors. The area of a parallelogram is a determinant. Here the sides are curved, but that only produces  $(du)^2$  and  $(dv)^2$ , which we ignore.

A change  $du$  gives one side of Figure 14.11—it is  $(\partial x / \partial u \mathbf{i} + \partial y / \partial u \mathbf{j}) du$ . Side 2 is  $(\partial x / \partial v \mathbf{i} + \partial y / \partial v \mathbf{j}) dv$ . The curving comes from second derivatives. The area (the cross product of the sides) is  $|J| du dv$ .

**Final remark** I can't resist looking at the change in the reverse direction. Now the rectangle is in  $xy$  and the parallelogram is in  $uv$ . In all formulas, exchange  $x$  for  $u$  and  $y$  for  $v$ :

$$\text{new } J = \begin{vmatrix} \partial u / \partial x & \partial u / \partial y \\ \partial v / \partial x & \partial v / \partial y \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{\text{old } J}. \quad (12)$$

This is exactly like  $du/dx = 1/(dx/du)$ . It is the derivative of the inverse function. The product of slopes is 1—stretch out, shrink back. From  $xy$  to  $uv$  we have 2 by 2 matrices, and the identity matrix  $I$  takes the place of 1:

$$\frac{dx}{du} \frac{du}{dx} = 1 \quad \text{becomes} \quad \begin{bmatrix} \partial x / \partial u & \partial x / \partial v \\ \partial y / \partial u & \partial y / \partial v \end{bmatrix} \begin{bmatrix} \partial u / \partial x & \partial u / \partial y \\ \partial v / \partial x & \partial v / \partial y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (13)$$

The first row times the first column is  $(\partial x / \partial u)(\partial u / \partial x) + (\partial x / \partial v)(\partial v / \partial x) = \partial x / \partial x = 1$ . The first row times the second column is  $(\partial x / \partial u)(\partial u / \partial y) + (\partial x / \partial v)(\partial v / \partial y) = \partial x / \partial y = 0$ . **The matrices are inverses of each other.** The determinants of a matrix and its inverse obey our rule: old  $J$  times new  $J = 1$ . Those  $J$ 's cannot be zero, just as  $dx/du$  and  $du/dx$  were not zero. (Inverse functions increase steadily or decrease steadily.)

In two dimensions, an area  $dx dy$  goes to  $J du dv$  and comes back to  $dx dy$ .

## 14.2 EXERCISES

### Read-through questions

We change variables to improve the a of integration. The disk  $x^2 + y^2 \leq 9$  becomes the rectangle  $0 \leq r \leq \underline{b}$ ,  $0 \leq \theta \leq \underline{c}$ . The inner limits on  $\iint dy dx$  are  $y = \pm \underline{d}$ . In polar coordinates this area integral becomes  $\iint \underline{e} = \underline{f}$ .

A polar rectangle has sides  $dr$  and g. Two sides are not h but the angles are still i. The area between the circles  $r = 1$  and  $r = 3$  and the rays  $\theta = 0$  and  $\theta = \pi/4$  is j. The integral  $\iint x dy dx$  changes to  $\iint \underline{k}$ . This is the l around the m axis. Then  $\bar{x}$  is the ratio n. This is the  $x$  coordinate of the o, and it is the p value of  $x$ .

In a rotation through  $\alpha$ , the point that reaches  $(u, v)$  starts at  $x = u \cos \alpha - v \sin \alpha$ ,  $y = \underline{q}$ . A rectangle in the  $uv$  plane comes from a r in  $xy$ . The areas are s so the stretching factor is  $J = \underline{t}$ . This is the determinant of the matrix u containing  $\cos \alpha$  and  $\sin \alpha$ . The moment of inertia  $\iint x^2 dx dy$  changes to  $\iint \underline{v} du dv$ .

For single integrals  $dx$  changes to w  $du$ . For double integrals  $dx dy$  changes to  $J du dv$  with  $J = \underline{x}$ . The stretching factor  $J$  is the determinant of the 2 by 2 matrix y. The functions  $x(u, v)$  and  $y(u, v)$  connect an  $xy$  region  $R$  to a  $uv$  region  $S$ , and  $\iint_R dx dy = \iint_S \underline{z} = \text{area of } \underline{A}$ . For polar coordinates  $x = \underline{B}$ ,  $y = \underline{C}$ . For  $x = u$ ,  $y = u + 4v$  the 2 by 2 determinant is  $J = \underline{D}$ . A square in the  $uv$  plane comes from a E in  $xy$ . In the opposite direction the change has  $u = x$  and  $v = \frac{1}{4}(y - x)$  and a new  $J = \underline{F}$ . This  $J$  is constant because this change of variables is G.

In 1–12  $R$  is a pie-shaped wedge:  $0 \leq r \leq 1$  and  $\pi/4 \leq \theta \leq 3\pi/4$ .

1 What is the area of  $R$ ? Check by integration in polar coordinates.

2 Find limits on  $\iint dy dx$  to yield the area of  $R$ , and integrate. *Extra credit:* Find limits on  $\iint dx dy$ .

3 Equation (1) with  $\alpha = \pi/4$  rotates  $R$  into the  $uv$  region  $S = \underline{\hspace{2cm}}$ . Find limits on  $\iint du dv$ .

4 Compute the centroid height  $\bar{y}$  of  $R$  by changing  $\iint y dx dy$  to polar coordinates. Divide by the area of  $R$ .

5 The region  $R$  has  $\bar{x} = 0$  because         . After rotation through  $\alpha = \pi/4$ , the centroid  $(\bar{x}, \bar{y})$  of  $R$  becomes the centroid          of  $S$ .

6 Find the centroid of any wedge  $0 \leq r \leq a$ ,  $0 \leq \theta \leq b$ .

7 Suppose  $R^*$  is the wedge  $R$  moved up so that the sharp point is at  $x = 0$ ,  $y = 1$ .

(a) Find limits on  $\iint dy dx$  to integrate over  $R^*$ .

(b) With  $x^* = x$  and  $y^* = y - 1$ , the  $xy$  region  $R^*$  corresponds to what region in the  $x^*y^*$  plane?

(c) After that change  $dx dy$  equals           $dx^* dy^*$ .

8 Find limits on  $\iint r dr d\theta$  to integrate over  $R^*$  in Problem 7.

9 The right coordinates for  $R^*$  are  $r^*$  and  $\theta^*$ , with  $x = r^* \cos \theta^*$  and  $y = r^* \sin \theta^* + 1$ .

(a) Show that  $J = r^*$  so  $dA = r^* dr^* d\theta^*$ .

(b) Find limits on  $\iint r^* dr^* d\theta^*$  to integrate over  $R^*$ .

10 If the centroid of  $R$  is  $(0, \bar{y})$ , the centroid of  $R^*$  is \_\_\_\_\_. The centroid of the circle with radius 3 and center  $(1, 2)$  is \_\_\_\_\_. The centroid of the upper half of that circle is \_\_\_\_\_.

11 The moments of inertia  $I_x, I_y, I_o$  of the original wedge  $R$  are \_\_\_\_\_.

12 The moments of inertia  $I_x, I_y, I_o$  of the shifted wedge  $R^*$  are \_\_\_\_\_.

**Problems 13–16 change four-sided regions to squares.**

13  $R$  has straight sides  $y = 2x, x = 1, y = 1 + 2x, x = 0$ . Locate its four corners and draw  $R$ . Find its area by geometry.

14 Choose  $a, b, c, d$  so that the change  $x = au + bv, y = cu + dv$  takes the previous  $R$  into  $S$ , the unit square  $0 \leq u \leq 1, 0 \leq v \leq 1$ . From the stretching factor  $J = ad - bc$  find the area of  $R$ .

15 The region  $R$  has straight sides  $x = 0, x = 1, y = 0, y = 2x + 3$ . Choose  $a, b, c$  so that  $x = u$  and  $y = au + bv + cv$  change  $R$  to the unit square  $S$ .

16 A nonlinear term  $uv$  was needed in Problem 15. Which regions  $R$  could change to the square  $S$  with a linear  $x = au + bv, y = cu + dv$ ?

**Draw the  $xy$  region  $R$  that corresponds in 17–22 to the  $uv$  square  $S$  with corners  $(0, 0), (1, 0), (0, 1), (1, 1)$ . Locate the corners of  $R$  and then its sides (like a jigsaw puzzle).**

17  $x = 2u + v, y = u + 2v$

18  $x = 3u + 2v, y = u + v$

19  $x = e^{2u+v}, y = e^{u+2v}$

20  $x = uv, y = v^2 - u^2$

21  $x = u, y = u(1 + u^2)$

22  $x = u \cos v, y = u \sin v$  (only three corners)

23 In Problems 17 and 19, compute  $J$  from equation (8). Then find the area of  $R$  from  $\iint_S |J| du dv$ .

24 In 18 and 20, find  $J = \partial(x, y)/\partial(u, v)$  and the area of  $R$ .

25 If  $R$  lies between  $x = 0$  and  $x = 1$  under the graph of  $y = f(x) > 0$ , then  $x = u, y = vf(u)$  takes  $R$  to the unit square  $S$ . Locate the corners of  $R$  and the point corresponding to  $u = \frac{1}{2}, v = 1$ . Compute  $J$  to prove what we know:

$$\text{area of } R = \int_0^1 f(x) dx = \int_0^1 \int_0^1 J du dv.$$

26 From  $r = \sqrt{x^2 + y^2}$  and  $\theta = \tan^{-1}(y/x)$ , compute  $\partial r/\partial x, \partial r/\partial y, \partial \theta/\partial x, \partial \theta/\partial y$ , and the determinant  $J = \partial(r, \theta)/\partial(x, y)$ . How is this  $J$  related to the factor  $r = \partial(x, y)/\partial(r, \theta)$  that enters  $r dr d\theta$ ?

27 Example 4 integrated  $e^{-x^2}$  from 0 to  $\infty$  (answer  $\sqrt{\pi}$ ). Also  $B = \int_0^1 e^{-x^2} dx$  leads to  $B^2 = \int_0^1 e^{-x^2} dx \int_0^1 e^{-y^2} dy$ . Change this double integral over the unit square to  $r$  and  $\theta$ — and find the limits on  $r$  that make exact integration impossible.

28 Integrate by parts to prove that the standard normal distribution  $p(x) = e^{-x^2/2}/\sqrt{2\pi}$  has  $\sigma^2 = \int_{-\infty}^{\infty} x^2 p(x) dx = 1$ .

29 Find the average distance from a point on a circle to the points inside. Suggestion: Let  $(0, 0)$  be the point and let  $0 \leq r \leq 2a \cos \theta, 0 \leq \theta \leq \pi$  be the circle (radius  $a$ ). The distance is  $r$ , so the average distance is  $\bar{r} = \iint \frac{r}{\text{area}} dA$ .

30 Draw the region  $R: 0 \leq x \leq 1, 0 \leq y < \infty$  and describe it with polar coordinates (limits on  $r$  and  $\theta$ ). Integrate  $\iint_R (x^2 + y^2)^{-3/2} dx dy$  in polar coordinates.

31 Using polar coordinates, find the volume under  $z = x^2 + y^2$  above the unit disk  $x^2 + y^2 \leq 1$ .

32 The end of Example 1 stated the moment of inertia  $\iint y^2 dA$ . Check that integration.

33 In the square  $-1 \leq x \leq 2, -2 \leq y \leq 1$ , where could you distribute a unit mass (with  $\iint \rho dx dy = 1$ ) to maximize

(a)  $\iint x^2 \rho dA$  (b)  $\iint y^2 \rho dA$  (c)  $\iint r^2 \rho dA$

34 True or false, with a reason:

(a) If the  $uv$  region  $S$  corresponds to the  $xy$  region  $R$ , then area of  $S$  = area of  $R$ .

(b)  $\iint x dA \leq \iint x^2 dA$

(c) The average value of  $f(x, y)$  is  $\iint f(x, y) dA$

(d)  $\int_{-\infty}^{\infty} x e^{-x^2} dx = 0$

(e) A polar rectangle has the same area as a straight-sided region with the same corners.

35 Find the mass of the tilted square in Example 1 if the density is  $\rho = xy$ .

36 Find the mass of the ring in Example 2 if the density is  $\rho = x^2 + y^2$ . This is the same as which moment of inertia with which density?

37 Find the polar moment of inertia  $I_o$  of the ring in Example 2 if the density is  $\rho = x^2 + y^2$ .

38 Give the following statement an appropriate name:  $\iint_R f(x, y) dA = f(P)$  times (area of  $R$ ), where  $P$  is a point in  $R$ . Which point  $P$  makes this correct for  $f = x$  and  $f = y$ ?

39 Find the  $xy$  coordinates of the top point in Figure 14.6a and check that it goes to  $(u, v) = (1, 1)$ .

## 14.3 Triple Integrals

At this point in the book, I feel I can speak to you directly. You can guess what triple integrals are like. Instead of a small interval or a small rectangle, there is a small box. Instead of length  $dx$  or area  $dx dy$ , the box has volume  $dV = dx dy dz$ . That is length times width times height. The goal is to put small boxes together (by integration). The main problem will be to discover the correct limits on  $x, y, z$ .

We could dream up more and more complicated regions in three-dimensional space. But I don't think you can see the method clearly without seeing the region clearly. In practice six shapes are the most important:

*box prism cylinder cone tetrahedron sphere.*

The box is easiest and the sphere may be the hardest (but no problem in spherical coordinates). Circular cylinders and cones fall in the middle, where  $xyz$  coordinates are possible but  $r\theta z$  are the best. I start with the box and prism and  $xyz$ .

**EXAMPLE 1** By triple integrals find the volume of a box and a prism (Figure 14.12).

$$\iiint_{\text{box}} dV = \int_{z=0}^1 \int_{y=0}^3 \int_{x=0}^2 dx dy dz \quad \text{and} \quad \iiint_{\text{prism}} dV = \int_{z=0}^1 \int_{y=0}^{3-3z} \int_{x=0}^2 dx dy dz$$

The inner integral for both is  $\int dx = 2$ . Lines in the  $x$  direction have length 2, cutting through the box and the prism. The middle integrals show the limits on  $y$  (since  $dy$  comes second):

$$\int_{y=0}^3 2 dy = 6 \quad \text{and} \quad \int_{y=0}^{3-3z} 2 dy = 6 - 6z.$$

After two integrations these are *areas*. The first area 6 is for a plane section through the box. The second area  $6 - 6z$  is cut through the prism. The shaded rectangle goes from  $y = 0$  to  $y = 3 - 3z$ —we needed and used the equation  $y + 3z = 3$  for the boundary of the prism. *At this point  $z$  is still constant!* But the area depends on  $z$ , because the prism gets thinner going upwards. The base area is  $6 - 6z = 6$ , the top area is  $6 - 6z = 0$ .

The outer integral multiplies those areas by  $dz$ , to give the volume of slices. They are horizontal slices because  $z$  came last. Integration adds up the slices to find the total volume:

$$\text{box volume} = \int_{z=0}^1 6 dz = 6 \quad \text{prism volume} = \int_{z=0}^1 (6 - 6z) dz = \left[ 6z - 3z^2 \right]_0^1 = 3.$$

The box volume  $2 \cdot 3 \cdot 1$  didn't need calculus. The prism is half of the box, so its volume was sure to be 3—but it is satisfying to see how  $6z - 3z^2$  gives the answer. Our purpose is to see how a triple integral works.

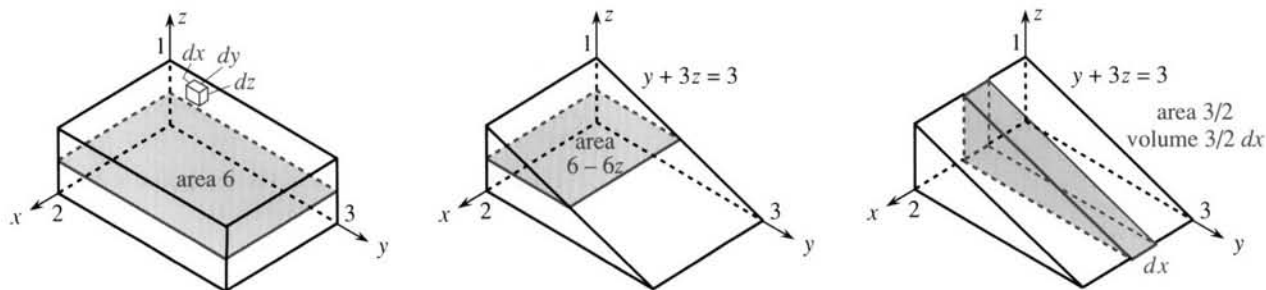


Fig. 14.12 Box with sides 2, 3, 1. The prism is half of the box: volume  $\int (6 - 6z) dz$  or  $\int \frac{3}{2} dx$ .

**Question** Find the prism volume in the order  $dz\,dy\,dx$  (*six orders are possible*).

**Answer** 
$$\int_0^2 \int_0^3 \int_0^{(3-y)/3} dz\,dy\,dx = \int_0^2 \int_0^3 \left(\frac{3-y}{3}\right) dy\,dx = \int_0^2 \frac{3}{2} dx = 3.$$

To find those limits on the  $z$  integral, follow a line in the  $z$  direction. It enters the prism at  $z=0$  and exits at the sloping face  $y+3z=3$ . That gives the upper limit  $z=(3-y)/3$ . It is the height of a thin stick as in Section 14.1. This section writes out  $\int dz$  for the height, but a quicker solution starts at the double integral.

What is the number  $\frac{3}{2}$  in the last integral? It is the **area of a vertical slice**, cut by a plane  $x=\text{constant}$ . The outer integral adds up slices.

$$\iiint f(x, y, z) \, dV \text{ is computed from three single integrals } \int \left[ \int \left[ \int f \, dx \right] dy \right] dz.$$

That step cannot be taken in silence—some basic calculus is involved. The triple integral is the limit of  $\sum f_i \Delta V$ , a sum over small boxes of volume  $\Delta V$ . Here  $f_i$  is any value of  $f(x, y, z)$  in the  $i$ th box. (In the limit, the boxes fit a curved region.) Now take those boxes in a *certain order*. Put them into lines in the  $x$  direction and put the lines of boxes into planes. The lines lead to the inner  $x$  integral, whose answer depends on  $y$  and  $z$ . The  $y$  integral combines the lines into planes. Finally the outer integral accounts for all planes and all boxes.

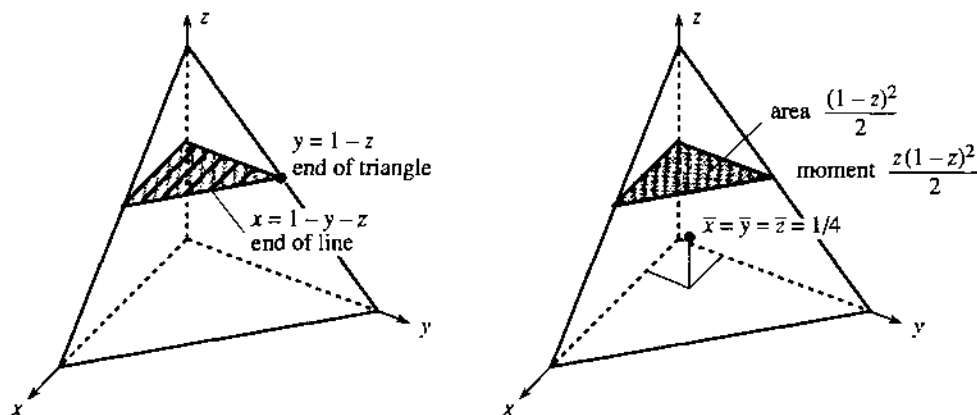
Example 2 is important because it displays more possibilities than a box or prism.

**EXAMPLE 2** Find the volume of a tetrahedron (4-sided pyramid). Locate  $(\bar{x}, \bar{y}, \bar{z})$ .

**Solution** A tetrahedron has four flat faces, all triangles. The fourth face in Figure 14.13 is on the plane  $x+y+z=1$ . A line in the  $x$  direction enters at  $x=0$  and exits at  $x=1-y-z$ . (The length depends on  $y$  and  $z$ . The equation of the boundary plane gives  $x$ .) Then those lines are put into plane slices by the  $y$  integral:

$$\int_{y=0}^{1-z} \int_{x=0}^{1-y-z} dx\,dy = \int_{y=0}^{1-z} (1-y-z) dy = \left[ y - \frac{1}{2}y^2 - zy \right]_0^{1-z} = \frac{1}{2}(1-z)^2.$$

What is this number  $\frac{1}{2}(1-z)^2$ ? **It is the area at height  $z$ .** The plane at that height slices out a right triangle, whose legs have length  $1-z$ . The area is correct, but look at the limits of integration. **If  $x$  goes to  $1-y-z$ , why does  $y$  go to  $1-z$ ?** Reason: We are assembling lines, not points. The figure shows a line at every  $y$  up to  $1-z$ .



**Fig. 14.13** Lines end at plane  $x+y+z=1$ . Triangles end at edge  $y+z=1$ . The average height is  $\bar{z} = \iiint z \, dV / \iiint dV$ .

Adding the slices gives the volume:  $\int_0^1 \frac{1}{2}(1-z)^2 dz = [\frac{1}{6}(z-1)^3]_0^1 = \frac{1}{6}$ . This agrees with  $\frac{1}{3}(\text{base times height})$ , the volume of a pyramid.

The height  $\bar{z}$  of the centroid is “ $z_{\text{average}}$ .” We compute  $\iiint z dV$  and divide by the volume. Each horizontal slice is multiplied by its height  $z$ , and the limits of integration don’t change:

$$\iiint z dV = \int_0^1 \int_0^{1-y} \int_0^{1-y-z} z dx dy dz = \int_0^1 \frac{z(1-z)^2}{2} dz = \frac{1}{24}.$$

This is quick because  $z$  is constant in the  $x$  and  $y$  integrals. Each triangular slice contributes  $z$  times its area  $\frac{1}{2}(1-z)^2$  times  $dz$ . Then the  $z$  integral gives the moment  $1/24$ . To find the *average* height, divide  $1/24$  by the volume:

$$\bar{z} = \text{height of centroid} = \frac{\iiint z dV}{\iiint dV} = \frac{1/24}{1/6} = \frac{1}{4}.$$

By symmetry  $\bar{x} = \frac{1}{4}$  and  $\bar{y} = \frac{1}{4}$ . The centroid is the point  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ . Compare that with  $(\frac{1}{3}, \frac{1}{3})$ , the centroid of the standard right triangle. Compare also with  $\frac{1}{2}$ , the center of the unit interval. There must be a five-sided region in four dimensions centered at  $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ .

For area and volume we meet another pattern. Length of standard interval is 1, area of standard triangle is  $\frac{1}{2}$ , volume of standard tetrahedron is  $\frac{1}{6}$ , hypervolume in four dimensions must be \_\_\_\_\_. The interval reaches the point  $x = 1$ , the triangle reaches the line  $x + y = 1$ , the tetrahedron reaches the plane  $x + y + z = 1$ . The four-dimensional region stops at the hyperplane \_\_\_\_\_ = 1.

**EXAMPLE 3** Find the volume  $\iiint dx dy dz$  inside the unit sphere  $x^2 + y^2 + z^2 = 1$ .

First question: What are the limits on  $x$ ? If a needle goes through the sphere in the  $x$  direction, where does it enter and leave? Moving in the  $x$  direction, the numbers  $y$  and  $z$  stay constant. The inner integral deals only with  $x$ . The smallest and largest  $x$  are at the boundary where  $x^2 + y^2 + z^2 = 1$ . This equation does the work—we solve it for  $x$ . Look at the limits on the  $x$  integral:

$$\text{volume of sphere} = \int_{-1}^1 \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} dx dy dz = \int_{-1}^1 \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} 2\sqrt{1-y^2-z^2} dy dz. \quad (1)$$

The limits on  $y$  are  $-\sqrt{1-z^2}$  and  $+\sqrt{1-z^2}$ . You can use algebra on the boundary equation  $x^2 + y^2 + z^2 = 1$ . But notice that  $x$  is gone! We want the smallest and largest  $y$ , for each  $z$ . It helps very much to draw the plane at height  $z$ , slicing through the sphere in Figure 14.14. The slice is a circle of radius  $r = \sqrt{1-z^2}$ . So the area is  $\pi r^2$ , which must come from the  $y$  integral:

$$\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} 2\sqrt{1-y^2-z^2} dy = \text{area of slice} = \pi(1-z^2). \quad (2)$$

I admit that I didn’t integrate. Is it cheating to use the formula  $\pi r^2$ ? I don’t think so. Mathematics is hard enough, and we don’t have to work blindfolded. The goal is understanding, and if you know the area then use it. Of course the integral of  $\sqrt{1-y^2-z^2}$  can be done if necessary—use Section 7.2.

The triple integral is down to a single integral. We went from one needle to a circle of needles and now to a sphere of needles. The volume is a sum of slices of area  $\pi(1-z^2)$ . The South Pole is at  $z = -1$ , the North Pole is at  $z = +1$ , and the integral

is the volume  $4\pi/3$  inside the unit sphere:

$$\int_{-1}^1 \pi(1-z^2)dz = \pi \left( z - \frac{1}{3}z^3 \right) \Big|_{-1}^1 = \frac{2}{3}\pi - \left( -\frac{2}{3}\pi \right) = \frac{4}{3}\pi. \quad (3)$$

**Question 1** A cone also has circular slices. How is the last integral changed?

**Answer** The slices of a cone have radius  $1-z$ . Integrate  $(1-z)^2$  not  $\sqrt{1-z^2}$ .

**Question 2** How does this compare with a circular cylinder (height 1, radius 1)?

**Answer** Now all slices have radius 1. Above  $z=0$ , a cylinder has volume  $\pi$  and a half-sphere has volume  $\frac{2}{3}\pi$  and a cone has volume  $\frac{1}{3}\pi$ .

For solids with equal surface area, the sphere has largest volume.

**Question 3** What is the average height  $\bar{z}$  in the cone and half-sphere and cylinder?

**Answer** 
$$\bar{z} = \frac{\int z(\text{slice area})dz}{\int (\text{slice area})dz} = \frac{1}{4} \quad \text{and} \quad \frac{3}{8} \quad \text{and} \quad \frac{1}{2}.$$

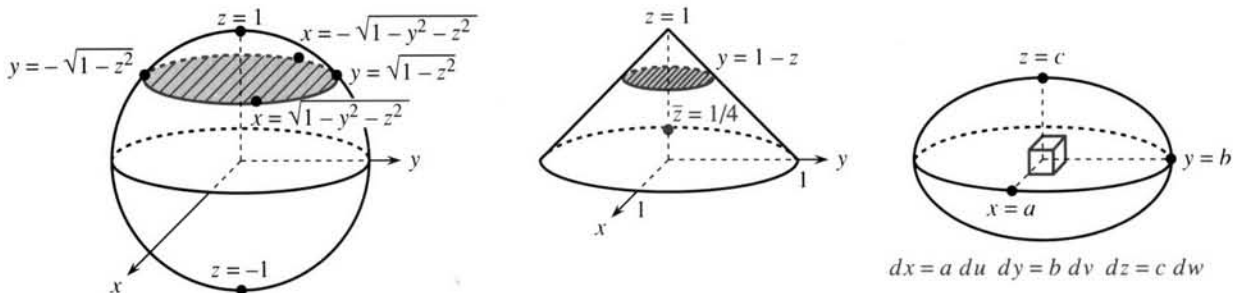


Fig. 14.14  $\int dx$  = length of needle,  $\iint dx dy$  = area of slice. Ellipsoid is a stretched sphere.

**EXAMPLE 4** Find the volume  $\iiint dx dy dz$  inside the ellipsoid  $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ .

The limits on  $x$  are now  $\pm \sqrt{1 - y^2/b^2 - z^2/c^2}$ . The algebra looks terrible. The geometry is better—all slices are ellipses. A **change of variable** is absolutely the best.

Introduce  $u = x/a$  and  $v = y/b$  and  $w = z/c$ . Then the outer boundary becomes  $u^2 + v^2 + w^2 = 1$ . In these new variables the shape is a sphere. The triple integral for a sphere is  $\iiint du dv dw = 4\pi/3$ . But what volume  $dV$  in  $xyz$  space corresponds to a small box with sides  $du$  and  $dv$  and  $dw$ ?

Every  $uvw$  box comes from an  $xyz$  box. The box is stretched with no bending or twisting. Since  $u$  is  $x/a$ , the length  $dx$  is  $a du$ . Similarly  $dy = b dv$  and  $dz = c dw$ . The volume of the  $xyz$  box (Figure 14.14) is  $dx dy dz = (abc) du dv dw$ . The **stretching factor**  $J = abc$  is a constant, and the volume of the ellipsoid is

$$\iiint_{\text{ellipsoid}} dx dy dz = \iiint_{\text{sphere}} (abc) du dv dw = \frac{4\pi}{3} abc. \quad (4)$$

You realize that this is special—other volumes are much more complicated. The sphere and ellipsoid are curved, but the small  $xyz$  boxes are straight. The next section introduces spherical coordinates, and we can finally write “good limits.” But then we need a different  $J$ .

## 14.3 EXERCISES

## Read-through questions

Six important solid shapes are a. The integral  $\iiint dx dy dz$  adds the volume b of small c. For computation it becomes d single integrals. The inner integral  $\int dx$  is the e of a line through the solid. The variables f and g are held constant. The double integral  $\iint dx dy$  is the h of a slice, with i held constant. Then the  $z$  integral adds up the volumes of j.

If the solid region  $V$  is bounded by the planes  $x=0$ ,  $y=0$ ,  $z=0$ , and  $x+2y+3z=1$ , the limits on the inner  $x$  integral are k. The limits on  $y$  are l. The limits on  $z$  are m. In the new variables  $u=x$ ,  $v=2y$ ,  $w=3z$ , the equation of the outer boundary is n. The volume of the tetrahedron in  $uvw$  space is o. From  $dx=du$  and  $dy=dv/2$  and  $dz=dw/3$ , the volume of an  $xyz$  box is  $dx dy dz =$  p  $du dv dw$ . So the volume of  $V$  is r.

To find the average height  $\bar{z}$  in  $V$  we compute s/t. To find the total mass in  $V$  if the density is  $\rho = e^z$  we compute the integral u. To find the average density we compute v/w. In the order  $\iiint dz dx dy$  the limits on the inner integral can depend on x. The limits on the middle integral can depend on y. The outer limits for the ellipsoid  $x^2 + 2y^2 + 3z^2 \leq 8$  are z.

1 For the solid region  $0 \leq x \leq y \leq z \leq 1$ , find the limits in  $\iiint dx dy dz$  and compute the volume.

2 Reverse the order in Problem 1 to  $\iiint dz dy dx$  and find the limits of integration. The four faces of this tetrahedron are the planes  $x=0$  and  $y=x$  and \_\_\_\_\_.

3 This tetrahedron and five others like it fill the unit cube. Change the inequalities in Problem 1 to describe the other five.

4 Find the centroid  $(\bar{x}, \bar{y}, \bar{z})$  in Problem 1.

Find the limits of integration in  $\iiint dx dy dz$  and the volume of solids 5–16. Draw a very rough picture.

5 A cube with sides of length 2, centered at  $(0, 0, 0)$ .

6 Half of that cube, the box above the  $xy$  plane.

7 Part of the same cube, the prism above the plane  $z=y$ .

8 Part of the same cube, above  $z=y$  and  $z=0$ .

9 Part of the same cube, above  $z=x$  and below  $z=y$ .

10 Part of the same cube, where  $x \leq y \leq z$ . What shape is this?

11 The tetrahedron bounded by planes  $x=0$ ,  $y=0$ ,  $z=0$ , and  $x+y+2z=2$ .

12 The tetrahedron with corners  $(0, 0, 0)$ ,  $(2, 0, 0)$ ,  $(0, 4, 0)$ ,  $(0, 0, 4)$ . First find the plane through the last three corners.

13 The part of the tetrahedron in Problem 11 below  $z=\frac{1}{2}$ .

14 The tetrahedron in Problem 12 with its top sliced off by the plane  $z=1$ .

15 The volume above  $z=0$  below the cone  $\sqrt{x^2+y^2}=1-z$ .

\*16 The tetrahedron in Problem 12, after it falls across the  $x$  axis onto the  $xy$  plane.

In 17–20 find the limits in  $\iiint dx dy dz$  or  $\iiint dz dy dx$ . Compute the volume.

17 A circular cylinder with height 6 and base  $x^2+y^2 \leq 1$ .

18 The part of that cylinder below the plane  $z=x$ . Watch the base. Draw a picture.

19 The volume shared by the cube (Problem 5) and cylinder.

20 The same cylinder lying along the  $x$  axis.

21 A cube is inscribed in a sphere: radius 1, both centers at  $(0, 0, 0)$ . What is the volume of the cube?

22 Find the volume and the centroid of the region bounded by  $x=0$ ,  $y=0$ ,  $z=0$ , and  $x/a + y/b + z/c = 1$ .

23 Find the volume and centroid of the solid  $0 \leq z \leq 4 - x^2 - y^2$ .

24 Based on the text, what is the volume inside  $x^2 + 4y^2 + 9z^2 = 16$ ? What is the “hypervolume” of the 4-dimensional pyramid that stops at  $x + y + z + w = 1$ ?

25 Find the partial derivatives  $\partial I / \partial x$ ,  $\partial I / \partial y$ ,  $\partial^2 I / \partial y \partial z$  of

$$I = \int_0^z \int_0^y \int_0^x dx dy dz \text{ and } I = \int_0^z \int_0^y \int_0^x f(x, y, z) dx dy dz.$$

26 Define the average value of  $f(x, y, z)$  in a solid  $V$ .

27 Find the moment of inertia  $\iiint l^2 dV$  of the cube  $|x| \leq 1$ ,  $|y| \leq 1$ ,  $|z| \leq 1$  when  $l$  is the distance to

(a) the  $x$  axis (b) the edge  $y=z=1$  (c) the diagonal  $x=y=z$ .

28 Add upper limits to produce the volume of a unit cube from small cubes:  $V = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n (\Delta x)^3 = 1$ .

\*29 Find the limit as  $\Delta x \rightarrow 0$  of  $\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \frac{2}{\Delta x} (\Delta x)^3$ .

30 The midpoint rule for an integral over the unit cube chooses the center value  $f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ . Which functions  $f = x^m y^n z^p$  are integrated correctly?

31 The trapezoidal rule estimates  $\int_0^1 \int_0^1 \int_0^1 f(x, y, z) dx dy dz$  as  $\frac{1}{8}$  times the sum of  $f(x, y, z)$  at 8 corners. This correctly integrates  $x^m y^n z^p$  for which  $m, n, p$ ?

32 Propose a 27-point “Simpson’s Rule” for integration over a cube. If many small cubes fill a large box, why are there only 8 new points per cube?



## 14.4 Cylindrical and Spherical Coordinates

Cylindrical coordinates are good for describing solids that are *symmetric around an axis*. The solid is three-dimensional, so there are three coordinates  $r, \theta, z$ :

$r$ : out from the axis     $\theta$ : around the axis     $z$ : along the axis.

This is a mixture of polar coordinates  $r\theta$  in a plane, plus  $z$  upward. You will not find  $r\theta z$  difficult to work with. Start with a cylinder centered on the  $z$  axis:

solid cylinder:  $0 \leq r \leq 1$     flat bottom and top:  $0 \leq z \leq 3$     half-cylinder:  $0 \leq \theta \leq \pi$

Integration over this half-cylinder is  $\int_0^3 \int_0^\pi \int_0^1 \underline{\hspace{1cm}} dr d\theta dz$ . These limits on  $r, \theta, z$  are especially simple. Two other axially symmetric solids are almost as convenient:

cone: integrate to  $r + z = 1$     sphere: integrate to  $r^2 + z^2 = R^2$

I would not use cylindrical coordinates for a box. Or a tetrahedron.

The integral needs one thing more—the volume  $dV$ . The movements  $dr$  and  $d\theta$  and  $dz$  give a “curved box” in  $xyz$  space, drawn in Figure 14.15c. The base is a polar rectangle, with area  $r dr d\theta$ . The new part is the height  $dz$ . **The volume of the curved box is  $r dr d\theta dz$ .** Then  $r$  goes in the blank space in the triple integral—the stretching factor is  $J = r$ . There are six orders of integration (we give two):

$$\text{volume} = \int_z \int_\theta \int_r r dr d\theta dz = \int_\theta \int_z \int_r r dr dz d\theta. \quad (1)$$

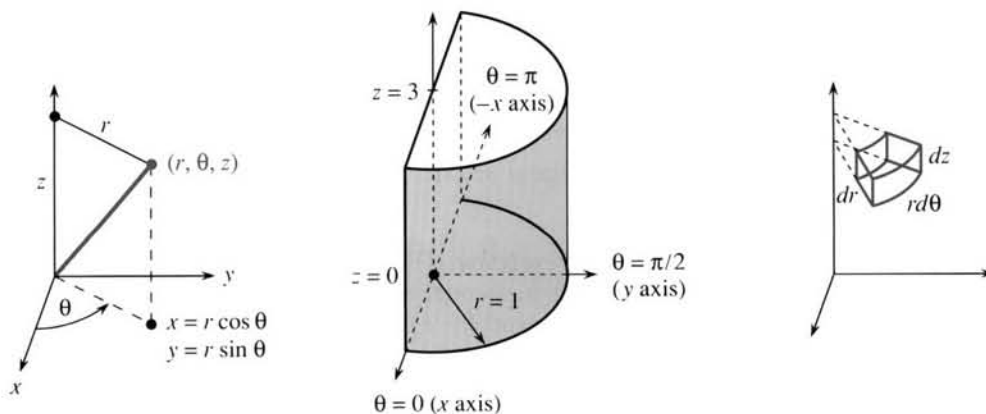


Fig. 14.15 Cylindrical coordinates for a point and a half-cylinder. Small volume  $r dr d\theta dz$ .

**EXAMPLE 1** (Volume of the half-cylinder). The integral of  $r dr$  from 0 to 1 is  $\frac{1}{2}$ . The  $\theta$  integral is  $\pi$  and the  $z$  integral is 3. The volume is  $3\pi/2$ .

**EXAMPLE 2** The surface  $r = 1 - z$  encloses the cone in Figure 14.16. Find its volume.

**First solution** Since  $r$  goes out to  $1 - z$ , the integral of  $r dr$  is  $\frac{1}{2}(1 - z)^2$ . The  $\theta$  integral is  $2\pi$  (a full rotation). Stop there for a moment.

We have reached  $\iint r dr d\theta = \frac{1}{2}(1 - z)^2 2\pi$ . This is the **area of a slice at height  $z$** . The slice is a circle, its radius is  $1 - z$ , its area is  $\pi(1 - z)^2$ . The  $z$  integral adds those slices to give  $\pi/3$ . That is correct, but it is not the only way to compute the volume.

**Second solution** Do the  $z$  and  $\theta$  integrals first. Since  $z$  goes up to  $1 - r$ , and  $\theta$  goes around to  $2\pi$ , those integrals produce  $\iint r \, dz \, d\theta = r(1 - r)2\pi$ . Stop again—this must be the area of something.

After the  $z$  and  $\theta$  integrals we have a *shell at radius  $r$* . The height is  $1 - r$  (the outer shells are shorter). This height times  $2\pi r$  gives the area around the shell. The choice between shells and slices is exactly as in Chapter 8. **Different orders of integration give different ways to cut up the solid.**

The volume of the shell is area times thickness  $dr$ . The volume of the complete cone is the integral of shell volumes:  $\int_0^1 r(1 - r)2\pi \, dr = \pi/3$ .

**Third solution** Do the  $r$  and  $z$  integrals first:  $\iint r \, dr \, dz = \frac{1}{6}$ . Then the  $\theta$  integral is  $\int \frac{1}{6} d\theta$ , which gives  $\frac{1}{6}$  times  $2\pi$ . This is the volume  $\pi/3$ —but what is  $\frac{1}{6} d\theta$ ?

The third cone is cut into wedges. The volume of a wedge is  $\frac{1}{6} d\theta$ . It is quite common to do the  $\theta$  integral last, especially when it just multiplies by  $2\pi$ . It is not so common to think of wedges.

**Question** Is the volume  $\frac{1}{6} d\theta$  equal to an area  $\frac{1}{6}$  times a thickness  $d\theta$ ?

**Answer** No! The triangle in the third cone has area  $\frac{1}{2}$  not  $\frac{1}{6}$ . Thickness is *never*  $d\theta$ .

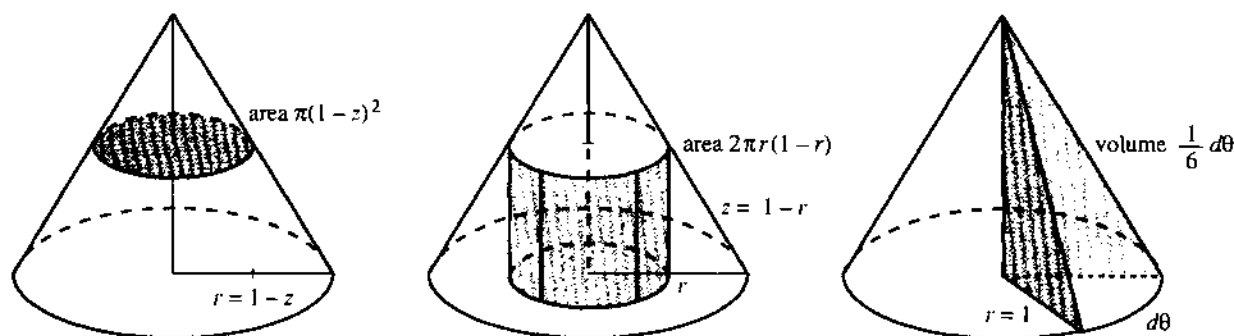


Fig. 14.16 A cone cut three ways: slice at height  $z$ , shell at radius  $r$ , wedge at angle  $\theta$ .

This cone is typical of a *solid of revolution*. The axis is in the  $z$  direction. The  $\theta$  integral yields  $2\pi$ , whether it comes first, second, or third. The  $r$  integral goes out to a radius  $f(z)$ , which is 1 for the cylinder and  $1 - z$  for the cone. The integral  $\iint r \, dr \, d\theta$  is  $\pi(f(z))^2 = \text{area of circular slice}$ . This leaves the  $z$  integral  $\int \pi(f(z))^2 \, dz$ . That is our old volume formula  $\int \pi(f(x))^2 \, dx$  from Chapter 8, where the slices were cut through the  $x$  axis.

**EXAMPLE 3** The *moment of inertia* around the  $z$  axis is  $\iiint r^2 \, dr \, d\theta \, dz$ . The extra  $r^2$  is (distance to axis)<sup>2</sup>. For the cone this triple integral is  $\pi/10$ .

**EXAMPLE 4** The *moment* around the  $z$  axis is  $\iiint r^2 \, dr \, d\theta \, dz$ . For the cone this is  $\pi/6$ . The *average distance  $\bar{r}$*  is (moment)/(volume) =  $(\pi/6)/(\pi/3) = \frac{1}{2}$ .

**EXAMPLE 5** A sphere of radius  $R$  has the boundary  $r^2 + z^2 = R^2$ , in cylindrical coordinates. The outer limit on the  $r$  integral is  $\sqrt{R^2 - z^2}$ . That is not acceptable in difficult problems. To avoid it we now change to coordinates that are natural for a sphere.

## SPHERICAL COORDINATES

The Earth is a solid sphere (or near enough). On its surface we use two coordinates—latitude and longitude. To dig inward or fly outward, there is a third coordinate—the distance  $\rho$  from the center. This Greek letter *rho* replaces  $r$  to avoid confusion with cylindrical coordinates. Where  $r$  is measured from the  $z$  axis,  $\rho$  is measured from the origin. Thus  $r^2 = x^2 + y^2$  and  $\rho^2 = x^2 + y^2 + z^2$ .

The angle  $\theta$  is the same as before. It goes from 0 to  $2\pi$ . It is the longitude, which increases as you travel east around the Equator.

The angle  $\phi$  is new. It equals 0 at the North Pole and  $\pi$  (not  $2\pi$ ) at the South Pole. It is the **polar angle**, measured down from the  $z$  axis. The Equator has a latitude of 0 but a polar angle of  $\pi/2$  (halfway down). Here are some typical shapes:

solid sphere (or ball):  $0 \leq \rho \leq R$

surface of sphere:  $\rho = R$

upper half-sphere:  $0 \leq \phi \leq \pi/2$

eastern half-sphere:  $0 \leq \theta \leq \pi$

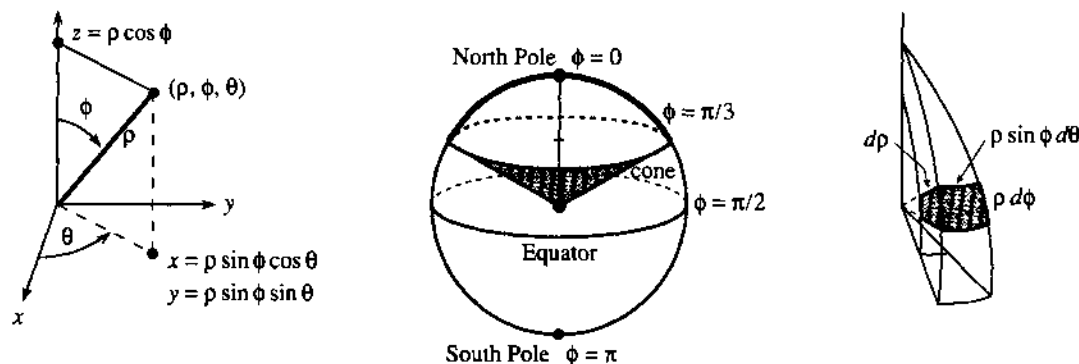


Fig. 14.17 Spherical coordinates  $\rho\phi\theta$ . The volume  $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$  of a spherical box.

The angle  $\phi$  is constant on a cone from the origin. It cuts the surface in a circle (Figure 14.17b), but not a great circle. The angle  $\theta$  is constant along a half-circle from pole to pole. The distance  $\rho$  is constant on each inner sphere, starting at the center  $\rho = 0$  and moving out to  $\rho = R$ .

**In spherical coordinates the volume integral is**  $\iiint \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$ . To explain that surprising factor  $J = \rho^2 \sin \phi$ , start with  $x = r \cos \theta$  and  $y = r \sin \theta$ . In spherical coordinates  $r$  is  $\rho \sin \phi$  and  $z$  is  $\rho \cos \phi$ —see the triangle in the figure. So substitute  $\rho \sin \phi$  for  $r$ :

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi. \quad (1)$$

Remember those two steps,  $\rho\phi\theta$  to  $r\theta z$  to  $xyz$ . We check that  $x^2 + y^2 + z^2 = \rho^2$ :

$$\rho^2(\sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta + \cos^2 \phi) = \rho^2(\sin^2 \phi + \cos^2 \phi) = \rho^2.$$

The volume integral is explained by Figure 14.17c. That shows a “spherical box” with right angles and curved edges. Two edges are  $d\rho$  and  $\rho d\phi$ . The third edge is horizontal. The usual  $r d\theta$  becomes  $\rho \sin \phi \, d\theta$ . Multiplying those lengths gives  $dV$ .

**The volume of the box is**  $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$ . This is a distance cubed, from  $\rho^2 d\rho$ .

**EXAMPLE 6** A solid ball of radius  $R$  has known volume  $V = \frac{4}{3}\pi R^3$ . Notice the limits:

$$\int_0^{2\pi} \int_0^\pi \int_0^R \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \left[ \frac{1}{3}\rho^3 \right]_0^R \left[ -\cos \phi \right]_0^\pi \left[ \theta \right]_0^{2\pi} = \left( \frac{4}{3}R^3 \right) (2)(2\pi).$$

**Question** What is the volume above the cone in Figure 14.17?

**Answer** The  $\phi$  integral stops at  $[-\cos \phi]_0^{\pi/3} = \frac{1}{2}$ . The volume is  $(\frac{4}{3}R^3)(\frac{1}{2})(2\pi)$ .

**EXAMPLE 7** The *surface area* of a sphere is  $A = 4\pi R^2$ . Forget the  $\rho$  integral:

$$A = \int_0^{2\pi} \int_0^\pi R^2 \sin \phi \, d\phi \, d\theta = R^2 \left[ -\cos \phi \right]_0^\pi \left[ \theta \right]_0^{2\pi} = R^2 (2)(2\pi).$$

*After those examples from geometry, here is the real thing from science.* I want to compute one of the most important triple integrals in physics—"the gravitational attraction of a solid sphere." For some reason Isaac Newton had trouble with this integral. He refused to publish his masterpiece on astronomy until he had solved it. I think he didn't use spherical coordinates—and the integral is not easy even now.

The answer that Newton finally found is beautiful. *The sphere acts as if all its mass were concentrated at the center.* At an outside point  $(0, 0, D)$ , the force of gravity is proportional to  $1/D^2$ . The force from a uniform solid sphere equals the force from a point mass, at every outside point  $P$ . That is exactly what Newton wanted and needed, to explain the solar system and to prove Kepler's laws.

Here is the difficulty. Some parts of the sphere are closer than  $D$ , some parts are farther away. The actual distance  $q$ , from the outside point  $P$  to a typical inside point, is shown in Figure 14.18. The *average* distance  $\bar{q}$  to all points in the sphere is not  $D$ . But what Newton needed was a different average, and by good luck or some divine calculus it works perfectly: *The average of  $1/q$  is  $1/D$ .* This gives the potential energy:

$$\text{potential at point } P = \iiint_{\text{sphere}} \frac{1}{q} dV = \frac{\text{volume of sphere}}{D}. \quad (2)$$

A small volume  $dV$  at the distance  $q$  contributes  $dV/q$  to the potential (Section 8.6, with density 1). The integral adds the contributions from the whole sphere. Equation (2) says that the potential at  $r = D$  is not changed when the sphere is squeezed to the center. The potential equals the whole volume divided by the single distance  $D$ .

Important point: The average of  $1/q$  is  $1/D$  and not  $1/\bar{q}$ . The average of  $\frac{1}{2}$  and  $\frac{1}{4}$  is not  $\frac{1}{3}$ . Smaller point: I wrote "sphere" where I should have written "ball." The sphere is solid:  $0 \leq \rho \leq R$ ,  $0 \leq \phi \leq \pi$ ,  $0 \leq \theta \leq 2\pi$ .

What about the force? For the small volume it is proportional to  $dV/q^2$  (this is the inverse square law). But *force is a vector*, pulling the outside point toward  $dV$ —not toward the center of the sphere. The figure shows the geometry and the symmetry. *We want the  $z$  component of the force.* (By symmetry the overall  $x$  and  $y$  components are zero.) The angle between the force vector and the  $z$  axis is  $\alpha$ , so for the  $z$  component we multiply by  $\cos \alpha$ . The total force comes from the integral that Newton discovered:

$$\text{force at point } P = \iiint_{\text{sphere}} \frac{\cos \alpha}{q^2} dV = \frac{\text{volume of sphere}}{D^2}. \quad (3)$$

I will compute the integral (2) and leave you the privilege of solving (3). I mean that word seriously. If you have come this far, you deserve the pleasure of doing what at

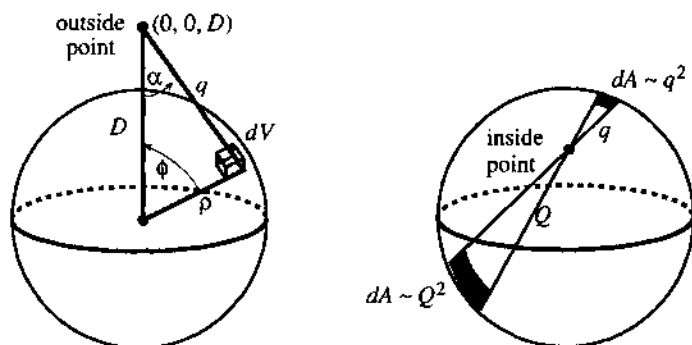


Fig. 14.18 Distance  $q$  from outside point to inside point. Distances  $q$  and  $Q$  to surface.

one time only Isaac Newton could do. Problem 26 offers a suggestion (just the law of cosines) but the integral is yours.

The law of cosines also helps with (2). For the triangle in the figure it gives  $q^2 = D^2 - 2\rho D \cos \phi + \rho^2$ . Call this whole quantity  $u$ . We do the surface integral first ( $d\phi$  and  $d\theta$  with  $\rho$  fixed). Then  $q^2 = u$  and  $q = \sqrt{u}$  and  $du = 2\rho D \sin \phi d\phi$ :

$$\int_0^{2\pi} \int_0^\pi \frac{\rho^2 \sin \phi d\phi d\theta}{q} = \int_{2\rho D}^{2\rho D + \rho^2} \frac{2\pi \rho^2 du}{2\rho D \sqrt{u}} = \left[ \frac{2\pi \rho}{D} \sqrt{u} \right]_{\phi=0}^{\phi=\pi}. \quad (4)$$

$2\pi$  came from the  $\theta$  integral. The integral of  $du/\sqrt{u}$  is  $2\sqrt{u}$ . Since  $\cos \phi = -1$  at the upper limit,  $u$  is  $D^2 + 2\rho D + \rho^2$ . The square root of  $u$  is  $D + \rho$ . At the lower limit  $\cos \phi = +1$  and  $u = D^2 - 2\rho D + \rho^2$ . This is another perfect square—its square root is  $D - \rho$ . The surface integral (4) with fixed  $\rho$  is

$$\iint \frac{dA}{q} = \frac{2\pi \rho}{D} [(D + \rho) - (D - \rho)] = \frac{4\pi \rho^2}{D}. \quad (5)$$

Last comes the  $\rho$  integral:  $\int_0^R 4\pi \rho^2 d\rho/D = \frac{4}{3}\pi R^3/D$ . This proves formula (2): **potential equals volume of the sphere divided by  $D$ .**

**Note 1** Physicists are also happy about equation (5). The average of  $1/q$  is  $1/D$  not only over the solid sphere but over each spherical shell of area  $4\pi\rho^2$ . The shells can have different densities, as they do in the Earth, and still Newton is correct. This also applies to the force integral (3)—**each separate shell acts as if its mass were concentrated at the center**. Then the final  $\rho$  integral yields this property for the solid sphere.

**Note 2** Physicists also know that force is minus the derivative of potential. The derivative of (2) with respect to  $D$  produces the force integral (3). Problem 27 explains this shortcut to equation (3).

**EXAMPLE 8** *Everywhere inside a hollow sphere the force of gravity is zero.*

When  $D$  is smaller than  $\rho$ , the lower limit  $\sqrt{u}$  in the integral (4) changes from  $D - \rho$  to  $\rho - D$ . That way the square root stays positive. This changes the answer in (5) to  $4\pi\rho^2/\rho$ , so the potential no longer depends on  $D$ . **The potential is constant inside the hollow shell.** Since the force comes from its derivative, the force is zero.

A more intuitive proof is in the second figure. The infinitesimal areas on the surface are proportional to  $q^2$  and  $Q^2$ . But the distances to those areas are  $q$  and  $Q$ , so the

forces involve  $1/q^2$  and  $1/Q^2$  (the inverse square law). Therefore the two areas exert equal and opposite forces on the inside point, and they cancel each other. The total force from the shell is zero.

I believe this zero integral is the reason that the inside of a car is safe from lightning. Of course a car is not a sphere. But electric charge distributes itself to keep the surface at constant potential. The potential stays constant inside—therefore no force. The tires help to prevent conduction of current (and electrocution of driver).

P.S. Don't just step out of the car. Let a metal chain conduct the charge to the ground. Otherwise you could be the conductor.

### CHANGE OF COORDINATES—STRETCHING FACTOR $J$

Once more we look to calculus for a formula. We need the volume of a small curved box in any  $uvw$  coordinate system. The  $r\theta z$  box and the  $\rho\phi\theta$  box have right angles, and their volumes were read off from the geometry (stretching factors  $J = r$  and  $J = \rho^2 \sin \phi$  in Figures 14.15 and 14.17). Now we change from  $xyz$  to other coordinates  $uvw$ —which are chosen to fit the problem.

Going from  $xy$  to  $uv$ , the area  $dA = J du dv$  was a 2 by 2 determinant. In three dimensions the determinant is 3 by 3. The matrix is always the “Jacobian matrix,” containing first derivatives. There were four derivatives from  $xy$  to  $uv$ , now there are nine from  $xyz$  to  $uvw$ .

**14C** Suppose  $x, y, z$  are given in terms of  $u, v, w$ . Then a small box in  $uvw$  space (sides  $du, dv, dw$ ) comes from a volume  $dV = J du dv dw$  in  $xyz$  space:

$$J = \begin{vmatrix} \partial x / \partial u & \partial x / \partial v & \partial x / \partial w \\ \partial y / \partial u & \partial y / \partial v & \partial y / \partial w \\ \partial z / \partial u & \partial z / \partial v & \partial z / \partial w \end{vmatrix} = \text{stretching factor } \frac{\partial(x, y, z)}{\partial(u, v, w)}. \quad (6)$$

The volume integral  $\iiint dx dy dz$  becomes  $\iiint |J| du dv dw$ , with limits on  $uvw$ .

Remember that a 3 by 3 determinant is the sum of six terms (Section 11.5). One term in  $J$  is  $(\partial x / \partial u)(\partial y / \partial v)(\partial z / \partial w)$ , along the main diagonal. This comes from pure stretching, and the other five terms allow for rotation. The best way to exhibit the formula is for spherical coordinates—where the nine derivatives are easy but the determinant is not:

**EXAMPLE 9** Find the factor  $J$  for  $x = \rho \sin \phi \cos \theta$ ,  $y = \rho \sin \phi \sin \theta$ ,  $z = \rho \cos \phi$ .

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix}.$$

The determinant has six terms, but two are zero—because of the zero in the corner. The other four terms are  $\rho^2 \sin \phi \cos^2 \phi \sin^2 \theta$  and  $\rho^2 \sin \phi \cos^2 \phi \cos^2 \theta$  and  $\rho^2 \sin^3 \phi \sin^2 \theta$  and  $\rho^2 \sin^3 \phi \cos^2 \theta$ . Add the first two (note  $\sin^2 \theta + \cos^2 \theta$ ) and separately add the second two. Then add the sums to reach  $J = \rho^2 \sin \phi$ .

Geometry already gave this answer. For most  $uvw$  variables, use the determinant.

## 14.4 EXERCISES

## Read-through questions

The three a coordinates are  $r\theta z$ . The point at  $x = y = z = 1$  has  $r =$  b,  $\theta =$  c,  $z =$  d. The volume integral is  $\iiint$  e. The solid region  $1 \leq r \leq 2$ ,  $0 \leq \theta \leq 2\pi$ ,  $0 \leq z \leq 4$  is a f. Its volume is g. From the  $r$  and  $\theta$  integrals the area of a h equals i. From the  $z$  and  $\theta$  integrals the area of a j equals k. In  $r\theta z$  coordinates the shapes of l are convenient, while m are not.

The three n coordinates are  $\rho\phi\theta$ . The point at  $x = y = z = 1$  has  $\rho =$  o,  $\phi =$  p,  $\theta =$  q. The angle  $\phi$  is measured from r.  $\theta$  is measured from s.  $\rho$  is the distance to t, where  $r$  was the distance to u. If  $\rho\phi\theta$  are known then  $x =$  v,  $y =$  w,  $z =$  x. The stretching factor  $J$  is a 3 by 3 y, and volume is  $\iiint$  z.

The solid region  $1 \leq \rho \leq 2$ ,  $0 \leq \phi \leq \pi$ ,  $0 \leq \theta \leq 2\pi$  is a A. Its volume is B. From the  $\phi$  and  $\theta$  integrals the area of a C at radius  $\rho$  equals D. Newton discovered that the outside gravitational attraction of a E is the same as for an equal mass located at F.

Convert the  $xyz$  coordinates in 1–4 to  $r\theta z$  and  $\rho\phi\theta$ .

- 1  $(D, 0, 0)$                       2  $(0, -D, 0)$   
3  $(0, 0, D)$  (watch  $\theta$ )        4  $(3, 4, 5)$

Convert the spherical coordinates in 5–7 to  $xyz$  and  $r\theta z$ .

- 5  $\rho = 4$ ,  $\phi = \pi/4$ ,  $\theta = -\pi/4$   
6  $\rho = 2$ ,  $\phi = \pi/3$ ,  $\theta = \pi/6$   
7  $\rho = 1$ ,  $\phi = \pi$ ,  $\theta =$  anything.  
8 Where does  $x = r$  and  $y = \theta$ ?

9 Find the polar angle  $\phi$  for the point with cylindrical coordinates  $r\theta z$ .

10 What are  $x(t)$ ,  $y(t)$ ,  $z(t)$  on the great circle from  $\rho = 1$ ,  $\phi = \pi/2$ ,  $\theta = 0$  with speed 1 to  $\rho = 1$ ,  $\phi = \pi/4$ ,  $\theta = \pi/2$ ?

From the limits of integration describe each region in 11–20 and find its volume. The inner integral has the inner limits.

- 11  $\int_{\theta=0}^{2\pi} \int_{r=0}^{1/\sqrt{2}} \int_{z=r}^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta$   
12  $\int_0^\pi \int_0^1 \int_0^{1+r^2} r \, dz \, dr \, d\theta$   
13  $\int_{\theta=0}^{2\pi} \int_{z=0}^1 \int_{r=0}^{2-z} r \, dr \, dz \, d\theta$   
14  $\int_0^\pi \int_0^\pi \int_0^\pi r \, d\theta \, dr \, dz$

$$15 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$16 \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec \phi}^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$17 \int_0^\pi \int_0^\pi \int_0^{\sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$18 \int_0^{2\pi} \int_0^{\pi/4} \int_1^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$19 \int_0^\pi \int_0^\pi \int_0^\pi \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$20 \int_0^1 \int_0^1 \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

21 Example 5 gave the volume integral for a sphere in  $r\theta z$  coordinates. What is the area of the circular slice at height  $z$ ? What is the area of the cylindrical shell at radius  $r$ ? Integrate over slices ( $dz$ ) and over shells ( $dr$ ) to reach  $4\pi R^3/3$ .

22 Describe the solid with  $0 \leq \rho \leq 1 - \cos \phi$  and find its volume.

23 A cylindrical tree has radius  $a$ . A saw cuts horizontally, ending halfway in at the  $x$  axis. Then it cuts on a sloping plane (angle  $\alpha$  with the horizontal), also ending at the  $x$  axis. What is the volume of the wedge that falls out?

24 Find the mass of a planet of radius  $R$ , if its density at each radius  $\rho$  is  $\delta = (\rho + 1)/\rho$ . Notice the infinite density at the center, but finite mass  $M = \iiint \delta \, dV$ . Here  $\rho$  is radius, not density.

25 For the cone out to  $r = 1 - z$ , the average distance from the  $z$  axis is  $\bar{r} = \frac{1}{2}$ . For the triangle out to  $r = 1 - z$  the average is  $\bar{r} = \frac{1}{3}$ . How can they be different when rotating the triangle produces the cone?

Problems 26–32, on the attraction of a sphere, use Figure 14.18 and the law of cosines  $q^2 = D^2 - 2\rho D \cos \phi + \rho^2 = u$ .

26 *Newton's achievement* Show that  $\iiint (\cos \alpha) dV/q^2$  equals volume/ $D^2$ . *One hint only:* Find  $\cos \alpha$  from a second law of cosines  $\rho^2 = D^2 - 2qD \cos \alpha + q^2$ . The  $\phi$  integral should involve  $1/q$  and  $1/q^3$ . Equation (2) integrates  $1/q$ , leaving  $\iiint dV/q^3$  still to do.

27 Compute  $\partial q / \partial D$  in the first cosine law and show from Figure 14.18 that it equals  $\cos \alpha$ . Then the derivative of equation (2) with respect to  $D$  is a shortcut to Newton's equation (3).

28 The lines of length  $D$  and  $q$  meet at the angle  $\alpha$ . Move the meeting point up by  $\Delta D$ . Explain why the other line stretches by  $\Delta q \approx \Delta D \cos \alpha$ . So  $\partial q / \partial D = \cos \alpha$  as before.

29 Show that the average distance is  $\bar{q} = 4R/3$ , from the North Pole ( $D = R$ ) to points on the Earth's surface ( $\rho = R$ ). To compute:  $\bar{q} = \iint q R^2 \sin \phi \, d\phi \, d\theta / (\text{area } 4\pi R^2)$ . Use the same substitution  $u$ .

30 Show as in Problem 29 that the average distance is  $\bar{q} = D + \frac{1}{3}\rho^2/D$ , from the outside point  $(0, 0, D)$  to points on the shell of radius  $\rho$ . Then integrate  $\iiint q \, dV$  and divide by  $4\pi R^3/3$  to find  $\bar{q}$  for the solid sphere.

31 In Figure 14.18b, it is not true that the areas on the surface are exactly proportional to  $q^2$  and  $Q^2$ . Why not? What happens to the second proof in Example 8?

32 For two solid spheres attracting each other (sun and planet), can we concentrate *both* spheres into point masses at their centers?

\*33 Compute  $\iiint \cos \alpha \, dV/q^3$  to find the force of gravity at  $(0, 0, D)$  from a cylinder  $x^2 + y^2 \leq a^2$ ,  $0 \leq z \leq h$ . Show from a figure why  $q^2 = r^2 + (D - z)^2$  and  $\cos \alpha = (D - z)/q$ .

34 A linear change of variables has  $x = au + bv + cw$ ,  $y = du + ev + fw$ , and  $z = gu + hv + iw$ . Write down the six terms in the determinant  $J$ . Three terms have minus signs.

35 A pure stretching has  $x = au$ ,  $y = bv$ , and  $z = cw$ . Find the 3 by 3 matrix and its determinant  $J$ . What is special about the  $xyz$  box in this case?

36 (a) The matrix in Example 9 has three columns. Find the lengths of those three vectors (sum of squares, then square root). Compare with the edges of the box in Figure 14.17.  
(b) Take the dot product of every column in  $J$  with every other column. Zero dot products mean right angles in the box. So  $J$  is the product of the column lengths.

37 Find the stretching factor  $J$  for cylindrical coordinates from the matrix of first derivatives.

38 Follow Problem 36 for cylindrical coordinates—find the length of each column in  $J$  and compare with the box in Figure 14.15.

39 Find the moment of inertia around the  $z$  axis of a spherical shell (radius  $\rho$ , density 1). The distance from the axis to a point on the shell is  $r =$  \_\_\_\_\_. Substitute for  $r$  to find

$$I(\rho) = \int_0^{2\pi} \int_0^\pi r^2 \rho^2 \sin \phi \, d\phi \, d\theta.$$

Divide by  $m r^2$  (which is  $4\pi \rho^4$ ) to compute the number  $J$  for a hollow ball in the rolling experiment of Section 8.5.

40 The moment of inertia of a solid sphere (radius  $R$ , density 1) adds up the hollow spheres of Problem 39:  $I = \int_0^R I(\rho) d\rho =$  \_\_\_\_\_. Divide by  $m R^2$  (which is  $\frac{4}{3}\pi R^5$ ) to find  $J$  in the rolling experiment. A solid ball rolls faster than a hollow ball because \_\_\_\_\_.

41 Inside the Earth, the force of gravity is proportional to the distance  $\rho$  from the center. Reason: The inner ball of radius  $\rho$  has mass proportional to \_\_\_\_\_ (assume constant density). The force is proportional to that mass divided by \_\_\_\_\_. The rest of the Earth (sphere with hole) exerts no force because \_\_\_\_\_.

42 Dig a tunnel through the center to Australia. Drop a ball in the tunnel at  $y = R$ ; Australia is  $y = -R$ . The force of gravity is  $-cy$  by Problem 41. Newton's law is  $my'' = -cy$ . What does the ball do when it reaches Australia?