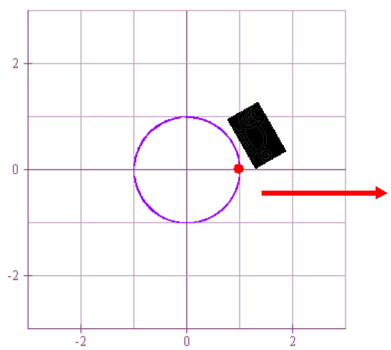
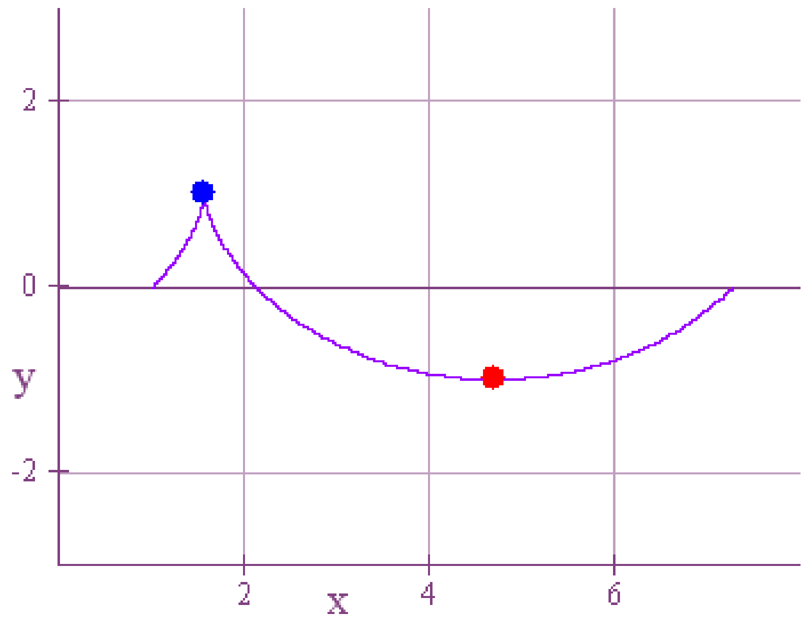
OTHER TRANSFORMATIONS – ANSWERS

1. Aliens who live in the 2-dimensional world of Flatland have built a giant 2-dimensional wheel of radius 1 mile, and a coordinate system is set up so that the center of the wheel is situated at the origin. Furthermore, the wheel is slowly moving to the right at a rate of 1 mile per hour, and at the same time a bug, beginning at coordinates $(1,0)$ is walking the perimeter of the wheel counterclockwise at a rate of 1 mile per hour. Find a vector-valued function in the form $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ that describes the bug's position after *t* hours, and find $\vec{r}'(t)$. Also, graph the bug's path over the interval $0 \le t \le 2\pi$, and find the values for *t* in that interval corresponding to when the bug reaches its highest elevation and when it reaches its lowest elevation, and find the bug's speed at each of those points. (NOTE: The wheel moves to the right, but does not rotate.)



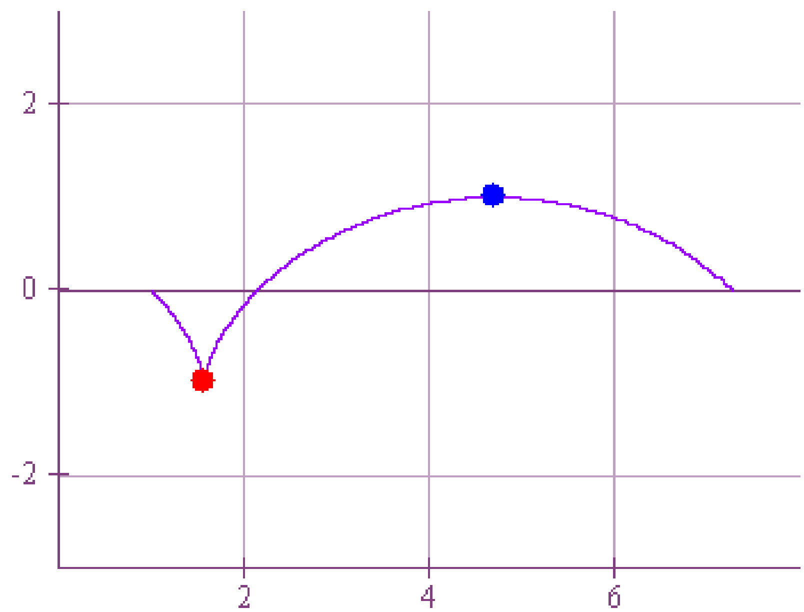
This type of problems is easily analyzed by thinking in terms of vector-valued functions. The circular motion as *t* goes from 0 to 2π is described by $\vec{r}_1(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$. Since the radius is 1 mile and the bug is walking at 1 mph, after an hour the angle will be 1 radian, and after *t* hours it will be *t* radians. The straight line motion to the right, however, is described by $\vec{r}_2(t) = t\,\hat{i}$. Now just add these two motions together and we get $\vec{r}(t) = \vec{r}_1(t) + \vec{r}_2(t) = (t + \cos t)\hat{i} + \sin t\,\hat{j}$. Clearly the highest elevation the bug will reach will be 1 and the lowest will be -1. These elevations will occur at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. The derivative is $\vec{r}'(t) = (1 - \sin t)\hat{i} + \cos t\,\hat{j}$, and the corresponding speeds are $\left\| \vec{r}'\left(\frac{\pi}{2}\right) \right\| = \sqrt{0^2 + 0^2} = 0$ and $\left\| \vec{r}'\left(\frac{3\pi}{2}\right) \right\| = \sqrt{2^2 + 0^2} = 2$. The graph below of $\vec{r}(t)$ is part of a *cycloid*.



2
0
y
-2
2
x
4
6

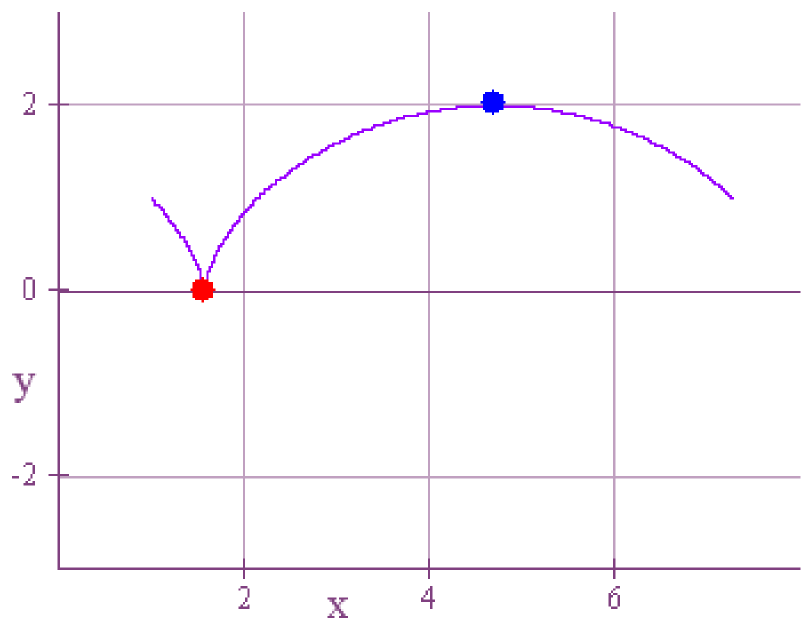
2. Repeat problem 1, but this time assume that the bug is standing still on the wheel, and the wheel is rotating <u>clockwise</u> at a rate of 1 radian per hour.

Since $\cos(-t) = \cos(t)$ and $\sin(-t) = -\sin(t)$, we can parametrize the circular path by $\vec{r}_1(t) = \cos t\,\hat{i} - \sin(t)\,\hat{j}$ for $0 \le t \le 2\pi$. Hence, our complete path is described by $\vec{r}(t) = \vec{r}_1(t) + \vec{r}_2(t) = (t + \cos t)\hat{i} - \sin(t)\,\hat{j}$, and again the path is part of a *cycloid*. The lowest elevation of -1 is reached when $t = \dfrac{\pi}{2}$, and the highest elevation of 1 is reached when $t = \dfrac{3\pi}{2}$. Also, $\vec{r}\,'(t) = (1 - \sin t)\hat{i} - \cos(-t)\,\hat{j} = (1 - \sin t)\hat{i} - \cos t\,\hat{j}$, and $\left\| \vec{r}\,'\left(\dfrac{\pi}{2}\right) \right\| = 0$, and $\left\| \vec{r}\,'\left(\dfrac{3\pi}{2}\right) \right\| = 2$.



3. Repeat problem 2, but this time assume that the center of the circle is at $(0,1)$ and the bug is initially at $(1,1)$

All we need to do is set up one more vector-valued function to describe our final path. In particular, shift everything up one unit by letting $\vec{r}_3(t)=\hat{j}$ and $\vec{r}(t)=\vec{r}_1(t)+\vec{r}_2(t)+\vec{r}_3(t)=(t+\cos t)\hat{i}+(1-\sin(t))\,\hat{j}$. Then $\vec{r}\,'(t)=(1-\sin t)\hat{i}-\cos t\,\hat{j}$, the lowest elevation is zero when $t=\dfrac{\pi}{2}$ and 2 when $t=\dfrac{3\pi}{2}$, and $\left\|\vec{r}\,'\left(\dfrac{\pi}{2}\right)\right\|=0$, and $\left\|\vec{r}\,'\left(\dfrac{3\pi}{2}\right)\right\|=2$.

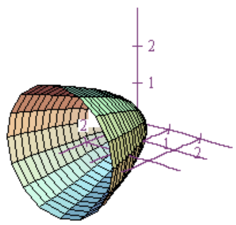


4. the x-axis

$\vec{r}(t,\theta) = t\hat{i} + \sqrt{t}\cos\theta\,\hat{j} + \sqrt{t}\sin\theta\,\hat{k}$

$0 \le t \le 3$

$0 \le \theta \le 2\pi$

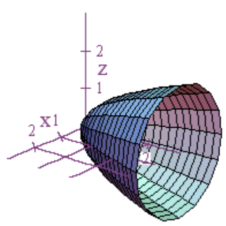


5. the y-axis

$\vec{r}(t,\theta) = \sqrt{t}\cos\theta\,\hat{i} + t\hat{j} + \sqrt{t}\sin\theta\,\hat{k}$

$0 \le t \le 3$

$0 \le \theta \le 2\pi$



6. the line $y = x$ in the xy-plane

$$\vec{r}(t,\theta) = \frac{\sqrt{t}\cos\theta + t}{\sqrt{2}}\hat{i} + \frac{-\sqrt{t}\cos\theta + t}{\sqrt{2}}\hat{j} + \sqrt{t}\sin\theta\hat{k}$$

$0 \le t \le 3$

$0 \le \theta \le 2\pi$