POLAR INTEGRALS - ANSWERS

Do the following by changing to polar coordinates.

1. Find the area of one petal of the rose $r = \cos 2\theta$.

$$\int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} r\,dr\,d\theta = 2\int_0^{\pi/4}\int_0^{\cos 2\theta} r\,dr\,d\theta = 2\int_0^{\pi/4} \left.\frac{r^2}{2}\right|_0^{\cos 2\theta} d\theta = \int_0^{\pi/4} \cos^2 2\theta\,d\theta$$

$$= \int_0^{\pi/4} \frac{1+\cos 4\theta}{2}\,d\theta = \left.\frac{\theta}{2} + \frac{\sin 4\theta}{8}\right|_0^{\pi/4} = \frac{\pi}{8}$$

2. Find the area of one petal of the rose $r = \sin 3\theta$

$$\int_0^{\pi/3}\int_0^{\sin 3\theta} r\,dr\,d\theta = \int_0^{\pi/3} \left.\frac{r^2}{2}\right|_0^{\sin 3\theta} d\theta = \int_0^{\pi/3} \frac{\sin^2 3\theta}{2}\,d\theta$$

$$= \int_0^{\pi/3} \frac{1-\cos 6\theta}{4}\,d\theta = \left.\frac{\theta}{4} - \frac{\sin 6\theta}{24}\right|_0^{\pi/3} = \frac{\pi}{12}$$

3. Prove that the area of a circle is πr^2 by evaluating $\iint_R dA$ where R is the disk $x^2 + y^2 \le r^2$.

$$\iint_R dA = \int_0^{2\pi}\int_0^{r} r\,dr\,d\theta = \int_0^{2\pi} \left.\frac{r^2}{2}\right|_0^{r} d\theta = \int_0^{2\pi} \frac{r^2}{2}\,d\theta = \left.\frac{\theta r^2}{2}\right|_0^{2\pi} = \pi r^2$$

4. Evaluate $\iint_R \left(x^2 + y^2\right) dA$ where R is the disk $x^2 + y^2 \le 4$.

$$\iint_R \left(x^2 + y^2\right) dA = \int_0^{2\pi}\int_0^{2} r^2\,r\,dr\,d\theta = \int_0^{2\pi} \left.\frac{r^4}{4}\right|_0^{2} d\theta = \int_0^{2\pi} 4\,d\theta = \left.4\theta\right|_0^{2\pi} = 8\pi$$

5. Evaluate $\iint_R \sqrt{x^2+y^2}\,dA$ where R is the disk $x^2+y^2 \le 1$.

$$\iint_R \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2}\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta = \int_0^{2\pi} \left.\frac{r^3}{3}\right|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}\,d\theta = \left.\frac{\theta}{3}\right|_0^{2\pi} = \frac{2\pi}{3}$$

6. Find the volume of the solid bounded above by $z = x^2+y^2+1$ and below by the disk $x^2+y^2 \le 1$.

$$\iint_R \left(x^2+y^2+1\right)dA = \int_0^{2\pi}\int_0^1 \left(r^2+1\right)r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(r^3+r\right)dr\,d\theta$$

$$= \int_0^{2\pi} \left.\left(\frac{r^4}{4}+\frac{r^2}{2}\right)\right|_0^1 d\theta = \int_0^{2\pi} \frac{3}{4}\,d\theta = \left.\frac{3\theta}{4}\right|_0^{2\pi} = \frac{3\pi}{2}$$

7. Find the volume inside the paraboloid $z = x^2+y^2$ for $0 \le z \le 4$.

$$\iint_R \left(4-(x^2+y^2)\right)dA = \int_0^{2\pi}\int_0^2 \left(4-r^2\right)r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left(4r-r^3\right)dr\,d\theta = \int_0^{2\pi} \left.\left(2r^2-\frac{r^4}{4}\right)\right|_0^2 d\theta$$

$$= \int_0^{2\pi} \left(8-4\right)d\theta = \int_0^{2\pi} 4\,d\theta = 4\theta\Big|_0^{2\pi} = 8\pi$$

8. Find the surface area of the portion of the paraboloid $z = 4-x^2-y^2$ that lies above the *xy*-plane

$$\text{surface area } = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\;dA = \iint_R \sqrt{\left(-2x\right)^2+\left(-2y\right)^2+1}\;dA$$

$$= \iint_R \sqrt{4x^2+4y^2+1}\;dA = \iint_R \sqrt{4r^2+1}\;r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left(4r^2+1\right)^{1/2} r\;dr\,d\theta$$

$$= \frac{1}{8}\int_0^{2\pi}\int_1^{17} u^{1/2}\;du\,d\theta = \int_0^{2\pi} \left.\frac{u^{3/2}}{12}\right|_1^{17} d\theta = \int_0^{2\pi} \frac{17^{3/2}-1}{12}\,d\theta = \frac{\pi}{6}\left(17^{3/2}-1\right) \approx 36.18$$