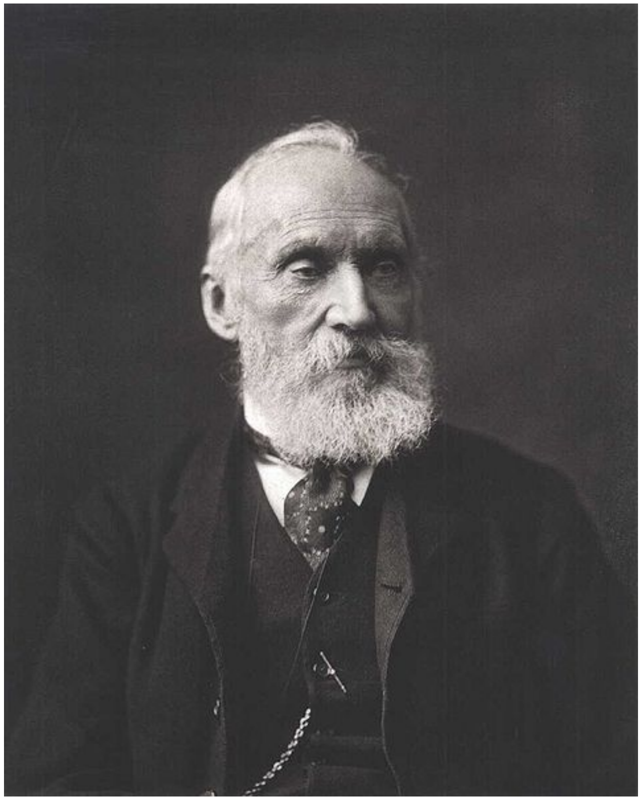
GREEN'S THEOREM,
STOKES' THEOREM,
AND
GAUSS' THEOREM



Recall Green's Theorem:

Theorem: Let *C* be a smooth, simple closed curve in the plane that is oriented counter-clockwise, and let *R* be the region bounded by *C*. If *P* and *Q* have continuous partial derivatives on an open region that contains *R*, then,

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

If $\vec{F} = P\hat{i} + Q\hat{j}$ and $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ is a smooth parametrization for C, then we can rewrite our equation as follows.

$$\text{Work } = \int_C \vec{F} \bullet d\vec{r} =$$

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

However, work is also a measure of circulation around a curve, and thus,

$$\text{Circulation} = \int_C \vec{F} \bullet T\, ds = \int_C \vec{F} \bullet d\vec{r}$$

$$= \int_C P\, dx + Q\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_R \left(curl\, \vec{F} \right) \bullet \hat{k}\, dA = \iint_R \left(\nabla \times \vec{F} \right) \bullet \hat{k}\, dA$$

This result is known as the 2-dimensional version of Stokes' Theorem.

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If we now go back to our parametrization for C, $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ with $a \le t \le b$, then the unit tangent is $T(t) = \dfrac{\vec{r}'(t)}{\|r'(t)\|} = \dfrac{x'(t)}{\|r'(t)\|}\hat{i} + \dfrac{y'(t)}{\|r'(t)\|}\hat{j}$, and the unit normal is $N(t) = \dfrac{y'(t)}{\|r'(t)\|}\hat{i} - \dfrac{x'(t)}{\|r'(t)\|}\hat{j}$.

Recall that the flux across the curve C produced by the force $\vec{F}$ is $\int_C \vec{F} \bullet N\, ds,$ and hence,

$$\text{Flux } = \int_C \vec{F} \bullet N\, ds = \int_a^b \left(\vec{F} \bullet N\right) \frac{ds}{dt} dt$$

$$= \int_a^b \left(P\hat{i} + Q\hat{j}\right) \bullet \left(\frac{y'(t)}{\left\|\vec{r}'(t)\right\|} \hat{i} - \frac{x'(t)}{\left\|\vec{r}'(t)\right\|} \hat{j} \right) \left\|\vec{r}'(t)\right\| dt$$

$$= \int_a^b \left(P\hat{i} + Q\hat{j}\right) \bullet \left(\frac{dy}{dt} \hat{i} - \frac{dx}{dt} \hat{j} \right) dt = \int_a^b \left(P\frac{dy}{dt} - Q\frac{dx}{dt} \right) dt$$

$$= \int_C Pdy - Qdx = \int_C -Qdx + Pdy = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

$$= \iint_R div\, \vec{F}\, dA = \iint_R \left(\nabla \bullet \vec{F}\right) dA$$

This result is known as the 2-dimension version of the Divergence Theorem or Gauss' Theorem.

$$\text{Flux } = \int_C \vec{F} \bullet N \, ds = \int_a^b \left(\vec{F} \bullet N \right) \frac{ds}{dt} dt$$

$$= \int_a^b \left(P\hat{i} + Q\hat{j} \right) \bullet \left(\frac{y'(t)}{\|\vec{r}'(t)\|} \hat{i} - \frac{x'(t)}{\|\vec{r}'(t)\|} \hat{j} \right) \|\vec{r}'(t)\| \, dt$$

$$= \int_a^b \left(P\hat{i} + Q\hat{j} \right) \bullet \left(\frac{dy}{dt} \hat{i} - \frac{dx}{dt} \hat{j} \right) dt = \int_a^b \left(P \frac{dy}{dt} - Q \frac{dx}{dt} \right) dt$$

$$= \int_C P dy - Q dx = \int_C -Q dx + P dy = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

$$= \iint_R div \, \vec{F} \, dA = \iint_R \left(\nabla \bullet \vec{F} \right) dA$$

To summarize, if $\vec{F} = P\hat{i} + Q\hat{j}$, C is a closed, counterclockwise oriented path paramtrized by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, T is the unit tangent vector, and N is the unit normal vector, then:

GREEN'S THEOREM: Work $= \int_C \vec{F} \bullet d\vec{r} = \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

STOKES' THEOREM: Circulation $= \int_C \vec{F} \bullet T\,ds = \int_C \vec{F} \bullet d\vec{r}$

$$= \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R \left(curl\,\vec{F} \right) \bullet \hat{k}\,dA = \iint_R \left(\nabla \times \vec{F} \right) \bullet \hat{k}\,dA$$

GAUSS' THEOREM: Flux $= \int_C \vec{F} \bullet N\,ds == \int_C Pdy - Qdx = \int_C -Qdx + Pdy$

$$= \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_R div\,\vec{F}\,dA = \iint_R \left(\nabla \bullet \vec{F} \right) dA$$

In higher dimensions, Stokes' Theorem and Gauss' Theorem (the Divergence Theorem) look like this:

STOKES' THEOREM: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then $\int_C \vec{F} \bullet d\vec{r} = \iint_S \left(curl\, F \bullet N \right) dS$.

GAUSS' THEOREM: Let V be a solid region whose boundary surface S has positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then $\iint_S \vec{F} \bullet N dS = \iiint_V div\, \vec{F}\, dV$

ENJOY!