ARC LENGTH

Suppose we have a curve C that is parametrized by a variable *t* as follows:

$$x = x(t)$$

$$y = y(t)$$

$$z = z(t)$$

$$a \leq t \leq b$$

We could also write this in vector as below:

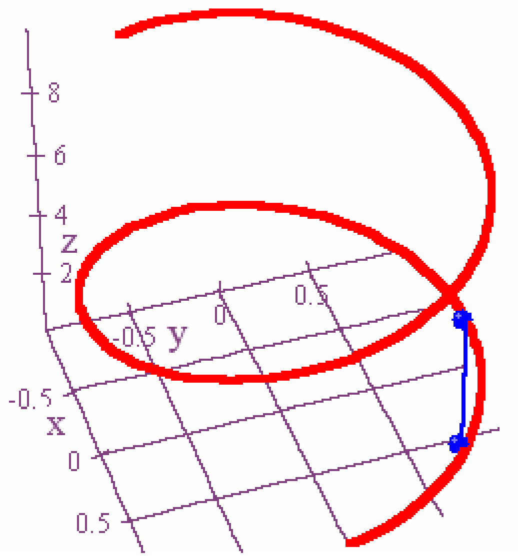
$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$a \le t \le b$$

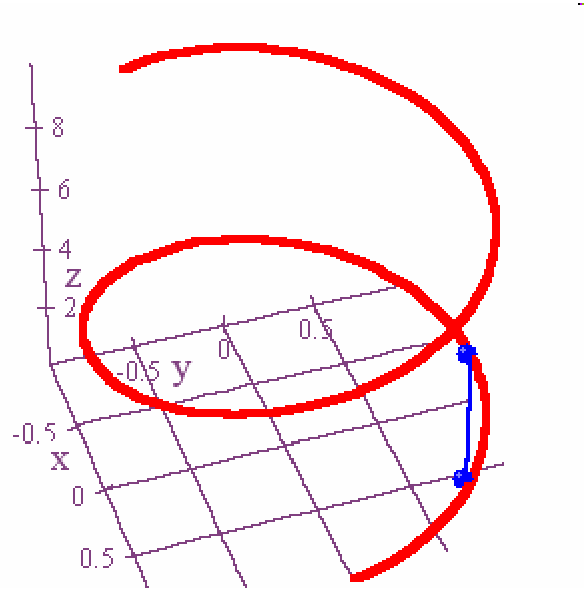
Below is the graph of the curve:

$$\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} + t\hat{k}$$

$$0 \le t \le 10$$

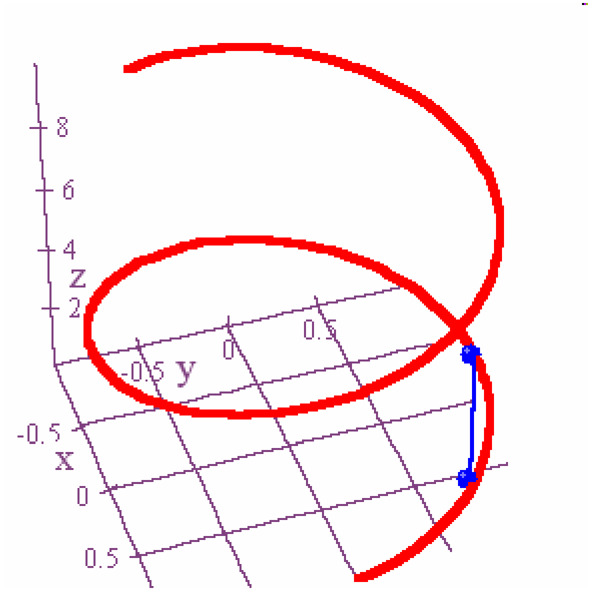


We can approximate the length of an arc of this curve by finding the length of the straight line segment connecting two points.

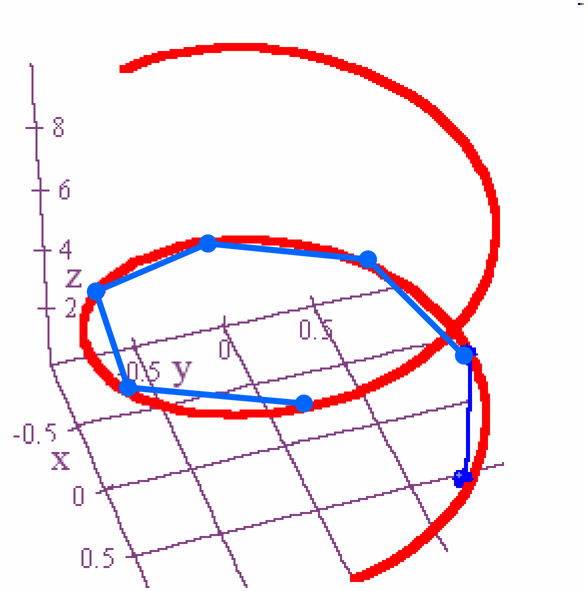


Using the distance formula, the length of this line segment is:

$$\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$



If we take several points along this curve and add up the lengths of the line segments connecting them, then we get the following:



$$Arc\ Length \approx \sum \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

Notice that:

$$\sum \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2} \cdot \Delta t$$

Hence:

$$Arc\,Length \approx \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2} \cdot \Delta t$$

And as a result:

$$Arc\,Length = \lim_{\Delta t \to 0} \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2} \cdot \Delta t$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b \left\| \vec{r}\,'(t) \right\| dt = \int_a^b \left\| \vec{v}(t) \right\| dt$$

We often like to express the arc length from *a* to *t* as:

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

As a consequence of The Fundamental Theorem of Calculus, if:

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

Then:

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \left\| \vec{r}\,'(t) \right\| = \left\| \vec{v}(t) \right\| = speed$$

Example: **Find the circumference of a circle of radius *r*.**

<u>Example:</u> **Find the circumference of a circle of radius *r*.**

$$\vec{r}(t) = r\cos(t)\hat{i} + r\sin(t)\hat{j}$$

$$0 \le t \le 2\pi$$

$$r = radius$$

Example: **Find the circumference of a circle of radius *r*.**

$$\vec{r}(t) = r\cos(t)\hat{i} + r\sin(t)\hat{j}$$

$$0 \le t \le 2\pi$$

$$r = radius$$

$$\vec{r}'(t) = -r\sin(t)\hat{i} + r\cos(t)\hat{j}$$

<u>Example:</u> **Find the circumference of a circle of radius *r*.**

$$\vec{r}(t) = r\cos(t)\hat{i} + r\sin(t)\hat{j}$$

$$0 \le t \le 2\pi$$

$$r = radius$$

$$\vec{r}'(t) = -r\sin(t)\hat{i} + r\cos(t)\hat{j}$$

$$Arc\,Length = \int_0^{2\pi} \left\|\vec{r}'(t)\right\| dt = \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\,dt$$

$$= \int_0^{2\pi} \sqrt{r^2}\,dt = \int_0^{2\pi} r\,dt = rt\Big|_0^{2\pi} = 2\pi r$$