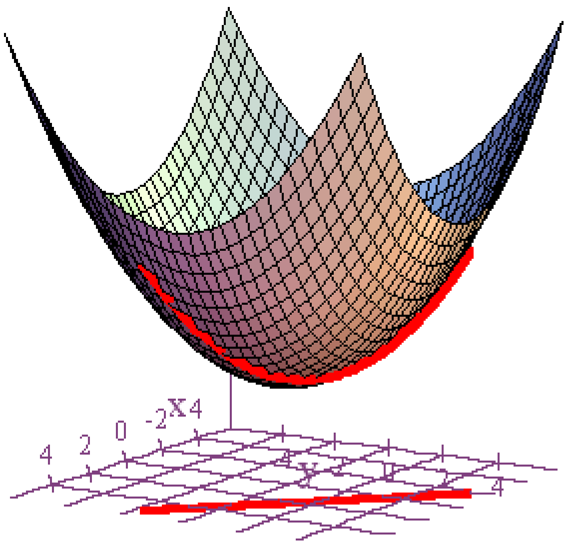
CONSTRAINED OPTIMIZATION

EXAMPLE 1: Suppose $z = f(x, y) = x^2 + y^2 + 20$, and our constraint curve is $x + y = 3$. Find the minimum value of $z = f(x, y)$ on this curve.



$$z = x^2 + y^2 + 20$$

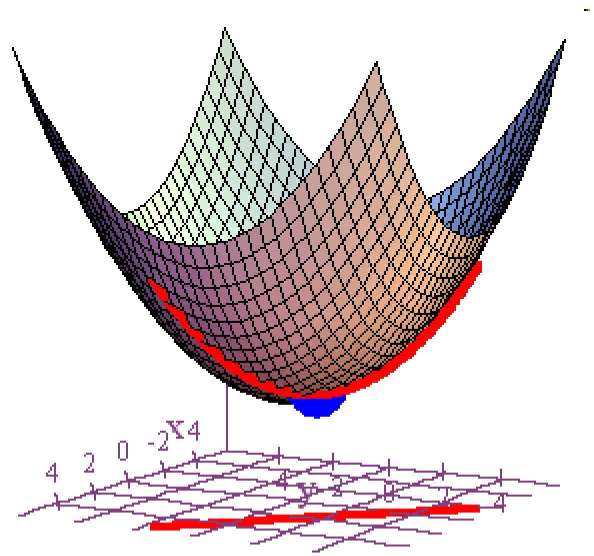
$$g = x + y$$

$$\begin{matrix} z_x = \lambda g_x \\ z_y = \lambda g_y \end{matrix} \Rightarrow \begin{matrix} 2x = \lambda \\ 2y = \lambda \end{matrix} \Rightarrow \begin{matrix} x = \dfrac{\lambda}{2} \\ y = \dfrac{\lambda}{2} \end{matrix}$$

$$x + y = 3 \Rightarrow \frac{\lambda}{2} + \frac{\lambda}{2} = 3 \Rightarrow \lambda = 3 \Rightarrow \begin{matrix} x = 3/2 \\ y = 3/2 \end{matrix}$$

$$f\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{9}{4} + \frac{9}{4} + 20 = \frac{49}{2} = 24.5$$

The minimum point is $\left(\frac{3}{2}, \frac{3}{2}, \frac{49}{2}\right) = (1.5, 1.5, 24.5)$.

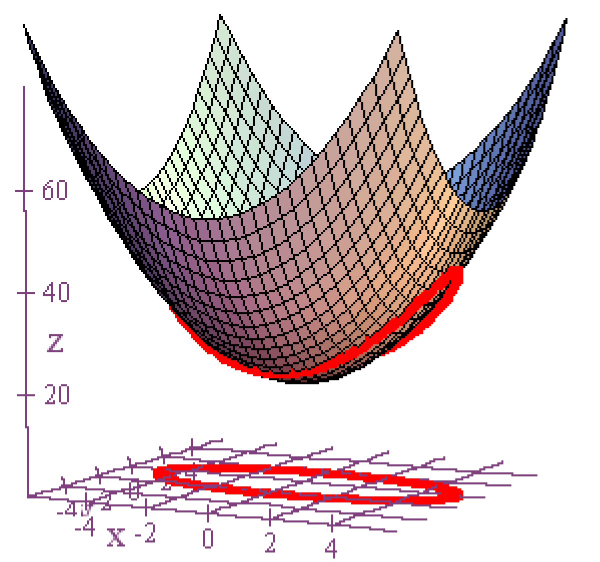


EXAMPLE 2: Find the maximum and minimum values of

$$z = f(x,y) = x^2 + y^2 + 20 \text{ on the ellipse } \frac{x^2}{25} + \frac{y^2}{4} = 1.$$

$$z = x^2 + y^2 + 20$$

$$g = \frac{x^2}{25} + \frac{y^2}{4}$$



$$z_x = 2x$$

$$g_x = \frac{2x}{25} \Rightarrow 2x = \frac{2\lambda x}{25} \Rightarrow x = 0 \text{ or } \lambda = 25$$

$$z_y = 2y$$

$$g_y = \frac{2y}{4} \Rightarrow 2y = \frac{2\lambda y}{4} \Rightarrow y = 0 \text{ or } \lambda = 4$$

$$z = x^2 + y^2 + 20$$

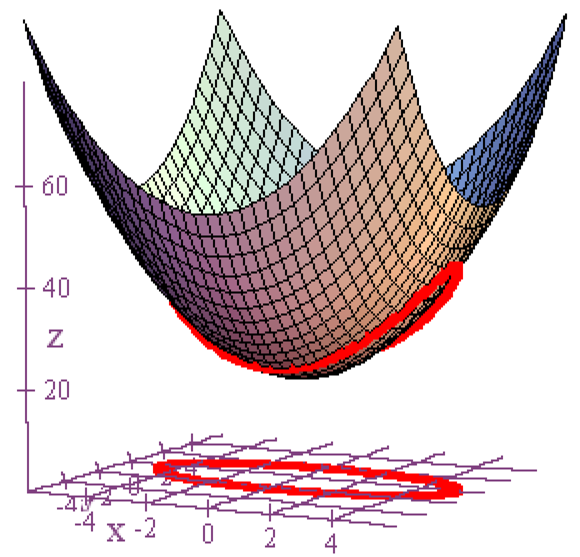
$$g = \frac{x^2}{25} + \frac{y^2}{4}$$

$$z_x = 2x$$

$$g_x = \frac{2x}{25} \Rightarrow 2x = \frac{2\lambda x}{25} \Rightarrow x = 0 \text{ or } \lambda = 25$$

$$z_y = 2y$$

$$g_y = \frac{2y}{4} \Rightarrow 2y = \frac{2\lambda y}{4} \Rightarrow y = 0 \text{ or } \lambda = 4$$



Hence, extreme values can only occur at the points $(0,-2), (0,2), (-5,0),$ and $(5,0)$.

$$z(0,-2) = 24 = \text{ min}$$

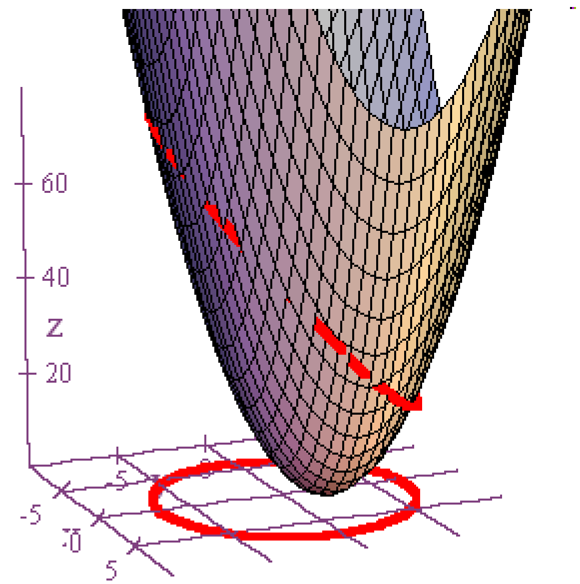
$$z(0,2) = 24 = \text{ min}$$

$$z(-5,0) = 45 = \text{ max}$$

$$z(5,0) = 45 = \text{ max}$$

EXAMPLE 3: Find the maximum and minimum values of

$z = f(x, y) = (x-1)^2 + (y-2)^2$ on the circle $x^2 + y^2 = 45.$

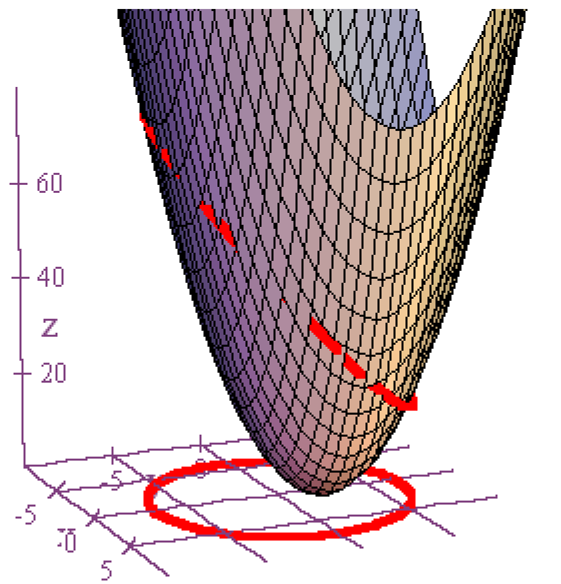


EXAMPLE 3: Find the maximum and minimum values of

$z = f(x, y) = (x-1)^2 + (y-2)^2$ on the circle $x^2 + y^2 = 45.$

$$z = (x-1)^2 + (y-2)^2$$

$$g = x^2 + y^2$$



$$\begin{matrix} z_x = 2x-2 \\ g_x = 2x \end{matrix} \Rightarrow 2x-2 = 2\lambda x \Rightarrow x-1 = \lambda x$$

$$\begin{matrix} z_y = 2y-4 \\ g_y = 2y \end{matrix} \Rightarrow 2y-4 = 2\lambda y \Rightarrow y-2 = \lambda y$$

$$z = (x-1)^2 + (y-2)^2$$

$$g = x^2 + y^2$$

$$x - 1 = \lambda x \Rightarrow x - \lambda x = 1 \Rightarrow x(1-\lambda) = 1 \Rightarrow x = \frac{1}{1-\lambda}$$

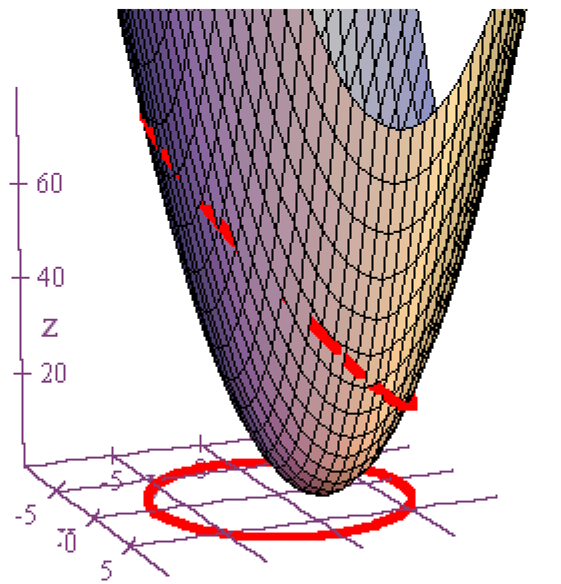
$$y - 2 = \lambda y \Rightarrow y - \lambda y = 2 \Rightarrow y(1-\lambda) = 2 \Rightarrow y = \frac{2}{1-\lambda}$$

$$x^2 + y^2 = 45 \Rightarrow \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{2}{1-\lambda}\right)^2 = 45$$

$$\Rightarrow \frac{5}{(1-\lambda)^2} = 45 \Rightarrow (1-\lambda)^2 = \frac{1}{9}$$

$$\Rightarrow \lambda^2 - 2\lambda + \frac{8}{9} = 0 \Rightarrow \lambda = \begin{cases} 4/3 \\ 2/3 \end{cases}$$

$$\Rightarrow x = -3 \,\&\, y = -6 \quad \text{or} \quad x = 3 \,\&\, y = 6$$



$$z(-3,-6) = (-4)^2 + (-8)^2 = 16 + 64 = 80 \leftarrow \text{ maximum}$$

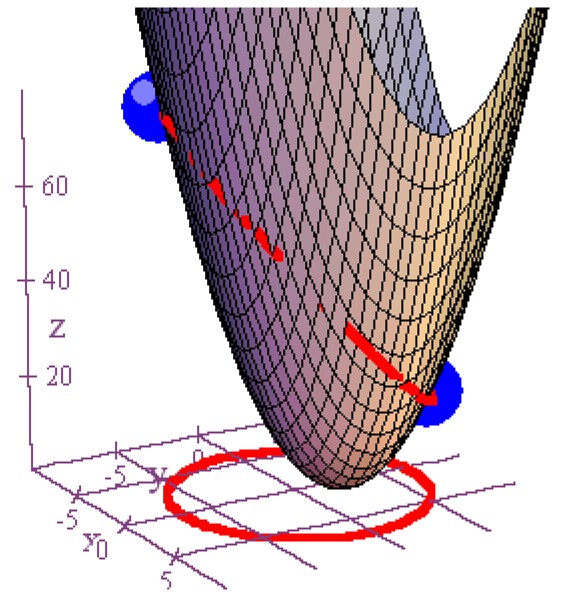
$$z(3,6) = 2^2 + 4^2 = 4 + 16 = 20 \leftarrow \text{ minimum}$$

$$x-1=\lambda x \Rightarrow x-\lambda x=1 \Rightarrow x(1-\lambda)=1 \Rightarrow x=\frac{1}{1-\lambda}$$

$$z=(x-1)^2+(y-2)^2$$

$$g=x^2+y^2$$

$$y-2=\lambda y \Rightarrow y-\lambda y=2 \Rightarrow y(1-\lambda)=2 \Rightarrow y=\frac{2}{1-\lambda}$$



$$x^2+y^2=45 \Rightarrow \left(\frac{1}{1-\lambda}\right)^2+\left(\frac{2}{1-\lambda}\right)^2=45$$

$$\Rightarrow \frac{5}{(1-\lambda)^2}=45 \Rightarrow (1-\lambda)^2=\frac{1}{9}$$

$$\Rightarrow \lambda^2-2\lambda+\frac{8}{9}=0 \Rightarrow \lambda=\begin{cases}4/3\\2/3\end{cases}$$

$$\Rightarrow x=-3 \,\&\, y=-6 \quad \text{or} \quad x=3 \,\&\, y=6$$

$$z(-3,-6)=(-4)^2+(-8)^2=16+64=80 \leftarrow \text{ maximum}$$

$$z(3,6)=2^2+4^2=4+16=20 \leftarrow \text{ minimum}$$