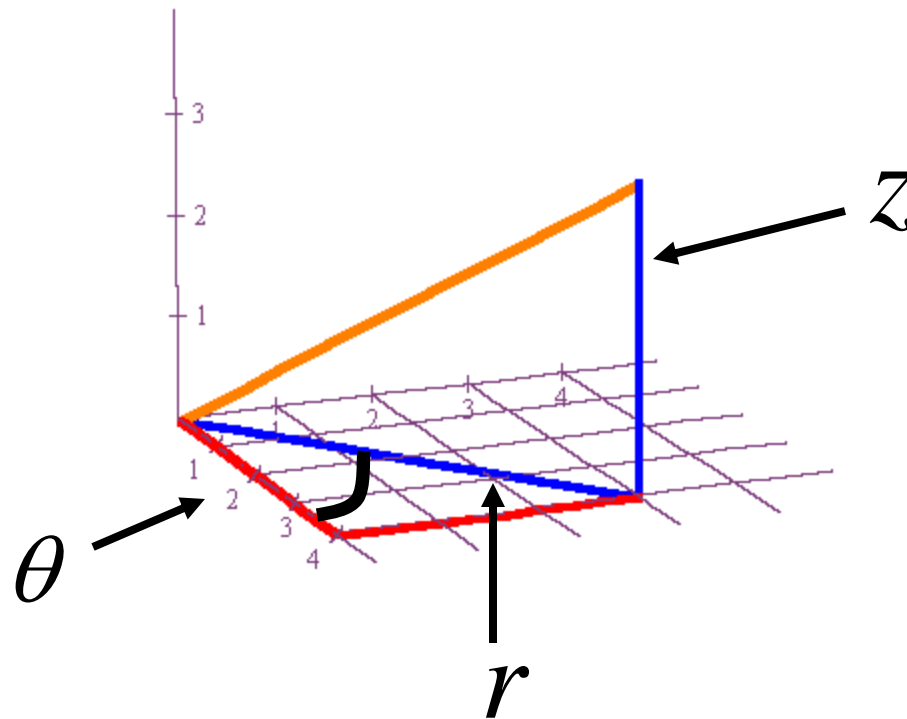


INTEGRALS IN POLAR AND CYLINDRICAL COORDINATES (r, θ, z)



$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

$$z = z$$

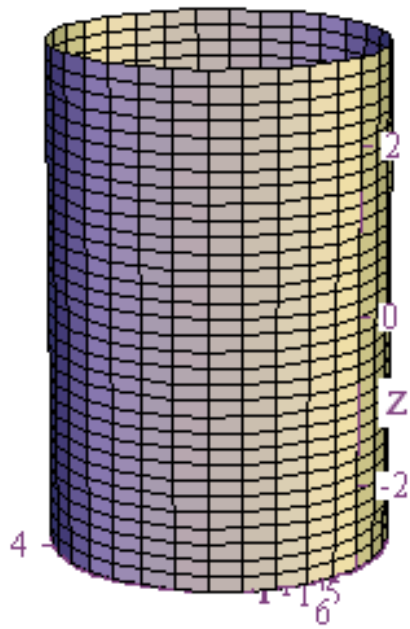
$$r^2 = x^2 + y^2$$

$$\tan \theta = \frac{y}{x}$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq r < \infty$$

$$-\infty < z < \infty$$

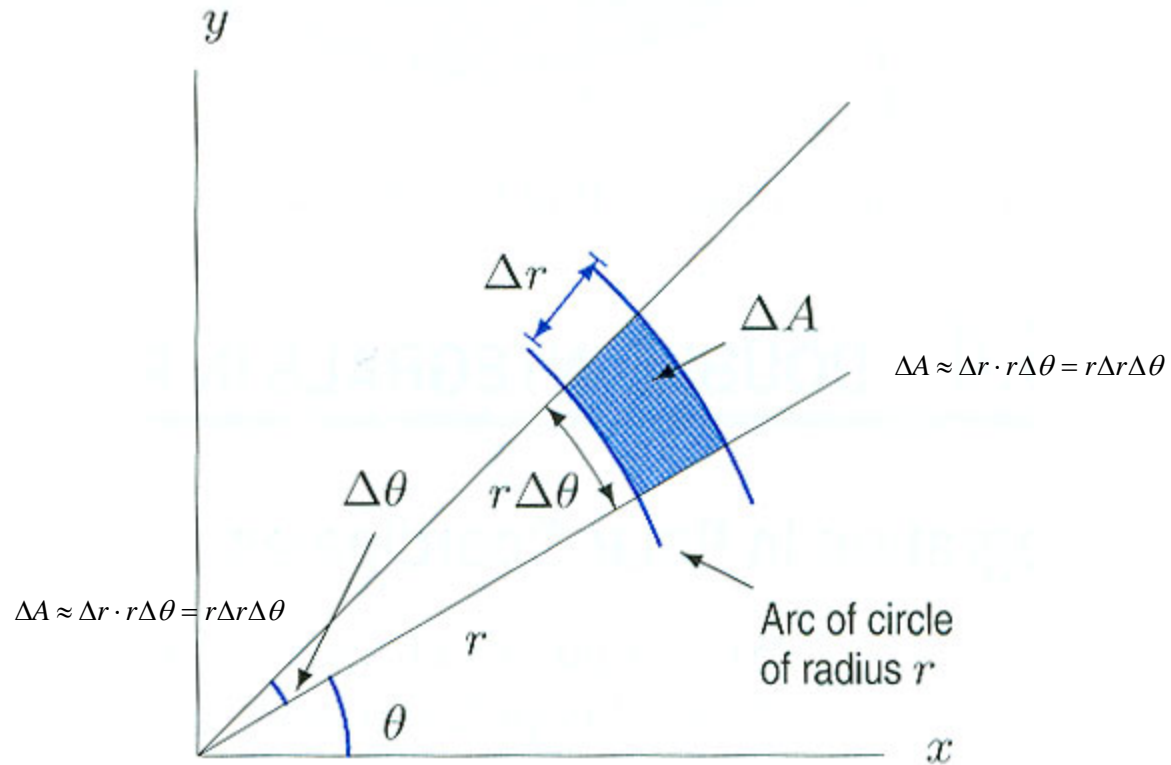


$$r = 2$$

$$0 \leq \theta \leq 2\pi$$

$$-3 \leq z \leq 3$$

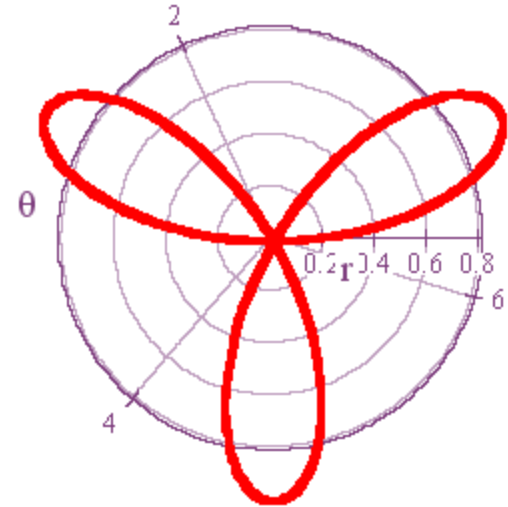
DOUBLE INTEGRALS IN POLAR COORDINATES



$$\Delta A \approx \Delta r \cdot r \Delta \theta = r \Delta r \Delta \theta$$

$$dA = r dr d\theta$$

EXAMPLE: Find the area of one petal of the three petal rose that is the graph of $r = \sin(3\theta)$.



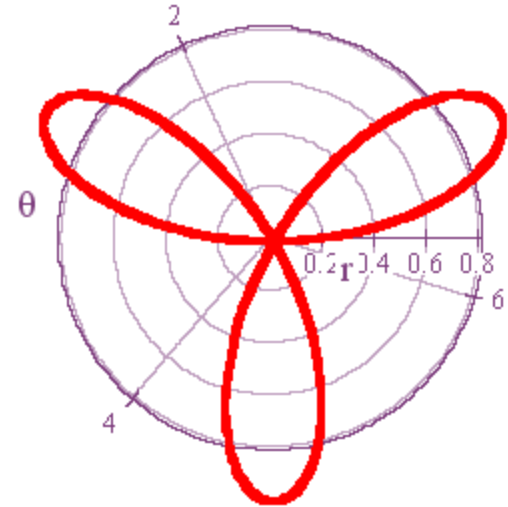
EXAMPLE: Find the area of one petal of the three petal rose that is the graph of $r = \sin(3\theta)$.

Let,

$$z = 1$$

$$0 \leq \theta \leq \frac{\pi}{3}$$

$$0 \leq r \leq \sin(3\theta)$$



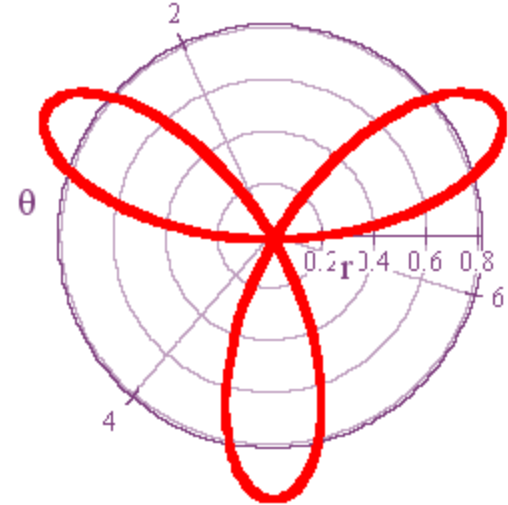
EXAMPLE: Find the area of one petal of the three petal rose that is the graph of $r = \sin(3\theta)$.

Let,

$$z = 1$$

$$0 \leq \theta \leq \frac{\pi}{3}$$

$$0 \leq r \leq \sin(3\theta)$$



$$\begin{aligned} \text{Area} &= \iint_R dA = \int_0^{\pi/3} \int_0^{\sin(3\theta)} r \, dr \, d\theta = \int_0^{\pi/3} \frac{r^2}{2} \Big|_0^{\sin(3\theta)} d\theta \\ &= \int_0^{\pi/3} \frac{\sin^2(3\theta)}{2} d\theta = \frac{1}{6} \int_0^{\pi} \sin^2 u \, du = \frac{1}{6} \int_0^{\pi} \frac{1 - \cos 2u}{2} du \\ &= \frac{1}{6} \left(\frac{u}{2} - \frac{\sin 2u}{4} \right) \Big|_0^{\pi} = \frac{\pi}{12} \end{aligned}$$

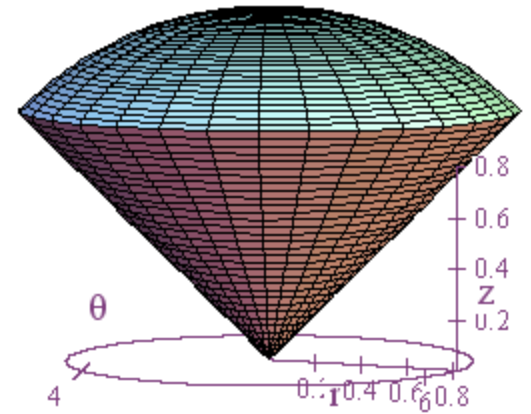
EXAMPLE: Find the integral of $z = \frac{1}{(x^2 + y^2)^{3/2}}$ on the region R corresponding to $0 \leq \theta \leq \pi/4$ and $1 \leq r \leq 2$.

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$$\begin{aligned} \iint_R \frac{1}{(x^2 + y^2)^{3/2}} dA &= \int_0^{\pi/4} \int_1^2 \frac{1}{(r^2)^{3/2}} r dr d\theta = \int_0^{\pi/4} \int_1^2 r^{-2} dr d\theta \\ &= \int_0^{\pi/4} -\frac{1}{r} \Big|_1^2 d\theta = \int_0^{\pi/4} \frac{1}{2} d\theta = \frac{\theta}{2} \Big|_0^{\pi/4} = \frac{\pi}{8} \end{aligned}$$

EXAMPLE: Find the volume of the ice cream cone defined by

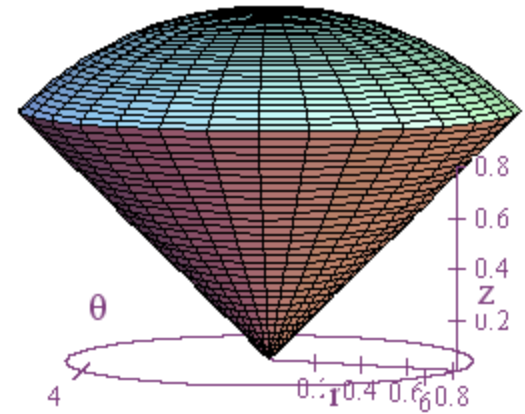
$$-1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \sqrt{x^2+y^2} \leq z \leq \sqrt{2-(x^2+y^2)}.$$



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$$-1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \sqrt{x^2+y^2} \leq z \leq \sqrt{2-(x^2+y^2)}.$$

$$\begin{aligned} \text{Volume} &= \iiint_V dV \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-(x^2+y^2)}} dz dy dx \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\sqrt{2-(x^2+y^2)} - \sqrt{x^2+y^2} \right) dy dx \end{aligned}$$



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$$-1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \sqrt{x^2+y^2} \leq z \leq \sqrt{2-(x^2+y^2)}.$$

$$\text{Volume} = \iiint_V dV$$

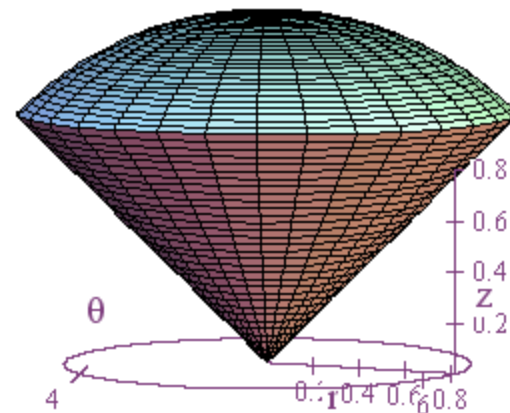
$$= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-(x^2+y^2)}} dz dy dx$$

$$= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\sqrt{2-(x^2+y^2)} - \sqrt{x^2+y^2} \right) dy dx$$

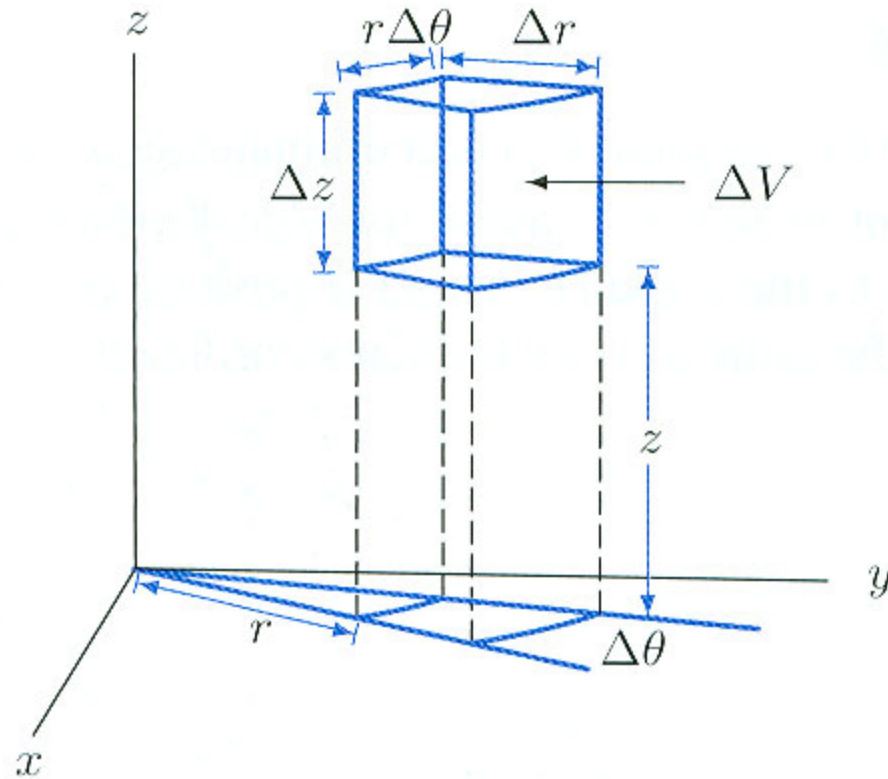
$$= \int_0^{2\pi} \int_0^1 \left(\sqrt{2-r^2} - r \right) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(r\sqrt{2-r^2} - r^2 \right) dr d\theta = \int_0^{2\pi} \left(\frac{-(2-r^2)^{3/2}}{3} - \frac{r^3}{3} \right) \Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{(2^{3/2} - 2)}{3} d\theta = \frac{\theta(2^{3/2} - 2)}{3} \Big|_0^{2\pi} = \frac{2\pi(2^{3/2} - 2)}{3} = \frac{4\pi}{3} (\sqrt{2} - 1)$$



TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

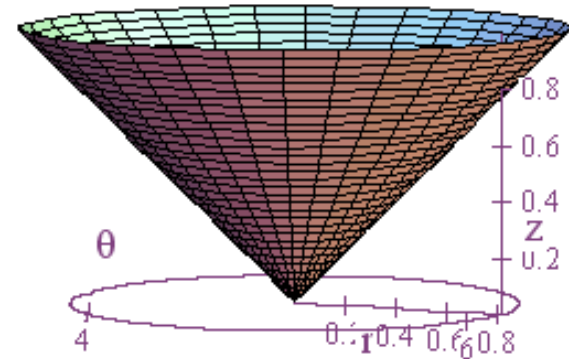


$$\Delta V \approx r\Delta\theta\Delta r\Delta z = r\Delta r\Delta\theta\Delta z$$

$$dV = r dr d\theta dz$$

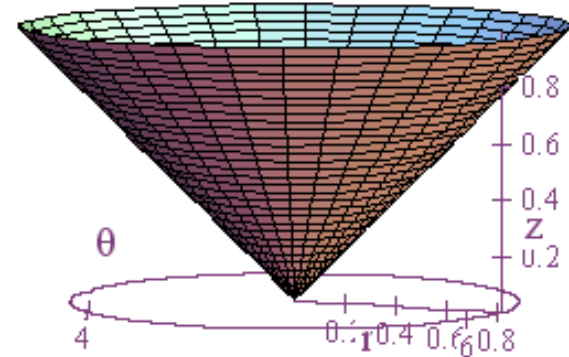
EXAMPLE: Find the volume beneath the cone defined by

$$z = \sqrt{x^2 + y^2} = r, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$



EXAMPLE: Find the volume beneath the cone defined by

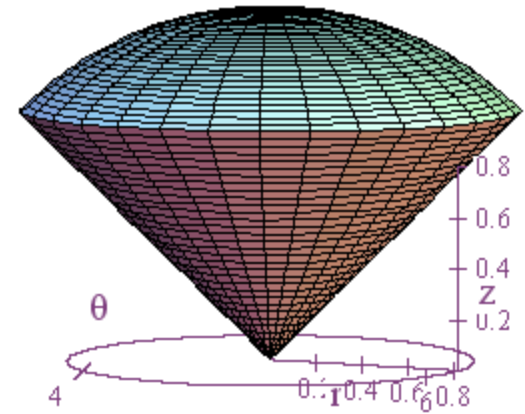
$$z = \sqrt{x^2 + y^2} = r, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$



$$\begin{aligned} \text{Volume} &= \iiint_V dV = \int_0^{2\pi} \int_0^1 \int_0^r r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r z \Big|_0^r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 r^2 \, dr \, d\theta = \int_0^{2\pi} \frac{r^3}{3} \Big|_0^1 \, d\theta = \int_0^{2\pi} \frac{1}{3} \, d\theta = \frac{\theta}{3} \Big|_0^{2\pi} = \frac{2\pi}{3}. \end{aligned}$$

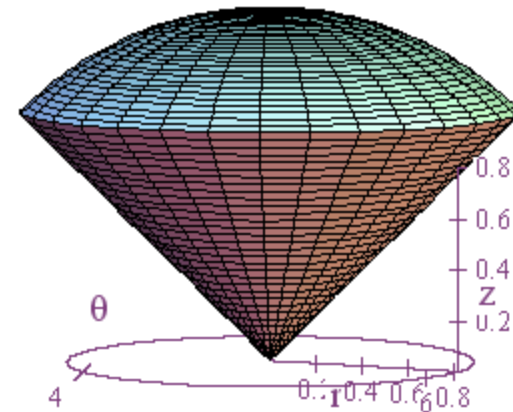
EXAMPLE: Find the volume of the ice cream cone defined by

$$r \leq z \leq \sqrt{2-r^2}, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi.$$



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$$r \leq z \leq \sqrt{2-r^2}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$



$$\begin{aligned} \text{Volume} &= \iiint_V dV = \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r z \Big|_r^{\sqrt{2-r^2}} \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 \left(r\sqrt{2-r^2} - r^2 \right) \, dr \, d\theta = \int_0^{2\pi} \left(\frac{-(2-r^2)^{3/2}}{3} - \frac{r^3}{3} \right) \Big|_0^1 \, d\theta \\ &= \int_0^{2\pi} \frac{(2^{3/2} - 2)}{3} \, d\theta = \frac{\theta(2^{3/2} - 2)}{3} \Big|_0^{2\pi} = \frac{2\pi(2^{3/2} - 2)}{3} = \frac{4\pi}{3} (\sqrt{2} - 1) \end{aligned}$$