THE EPSILON-DELTA DEFINITION OF A LIMIT

In the beginning …

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Leibniz and Newton created/discovered Calculus

Calculus soon became the mathematics of change.

Calculus studied rates of change (derivatives) and accumulated change (integrals).

However, the foundations rested originally on arguments that were very shaky.

Newton and Leibniz talked in terms of *infinitesimals*.

An infinitesimal is supposed to be a number that is greater than zero, but less than any positive number.

DUH?

Nonetheless, this illogic worked well in practice.

To find the rate of change of the function $f(x)=x^2$, move an infinitesimal distance dx away from x, and find the slope of the line through $(x, f(x))$ and $(x+dx, f(x+dx))$.

$$slope = \frac{f(x+dx) - f(x)}{(x+dx) - x} = \frac{(x+dx)^2 - x^2}{dx} = \frac{x^2 + 2xdx + dx^2 - x^2}{dx}$$

$$= \frac{2xdx + dx^2}{dx} = \frac{dx(2x+dx)}{dx} = 2x + dx$$

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Since *dx* is infinitesimally small, discard it and you get the correct result that the rate of change is *2x*.

In the 1700s, the theologian and philosopher Bishop Berkeley published a scathing critique of the illogical methods of calculus.

THE
ANALYST
OR, A
DISCOURSE

Addressed to an
Infidel Mathematician,
WHEREIN

It is examined whether the Object, Principles, and inferences of the modern analysis are more distinctly conceived, or more evidently deduced, than religious mysteries and points of faith.

THE ANALYST pointed to a need to develop a foundation for calculus that did not depend on infinitesimals.

To meet this need, the concept of a *limit* was formulated.

Instead of saying that we take a point that is infinitesimally close to *x*, we simply ask what happens as we let values get *arbitrarily close* to *x*.

In particular, we often want to know whether $f(x)$ gets closer to anything as our input values get closer to x. If so, then we call the value that $f(x)$ gets close to the *limit*.

However, the notion of “close,” while maybe intuitively clear, is not really well defined in the previous statement.

For a good definition of *limit* and what we mean by close, we need the epsilon-delta definition created by Karl Weierstrass in the 1800s.

Karl Weierstrass is known as the father of modern analysis. He started out as a high school teacher, but was later elevated to professor at the Technical University of Berlin as a result of the many brilliant papers he published.

He defined “closeness” in terms of being within a distance *epsilon* from some number.

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To say that the *limit* of *f(x)* as *x* approaches *a* is equal to *L* means that we can make the value of *f(x)* within a distance of *epsilon* units from *L* simply by making *x* within an appropriate distance of *delta* units from *x*.

We write this more formally as follows:

$$\lim_{x \to a} f(x) = L \textit{ if and only if for every } \varepsilon > 0, \textit{there exists } \delta > 0$$
$$\textit{such that if } 0 < \left| x - a \right| < \delta, \textit{then } \left| f(x) - L \right| < \varepsilon.$$

When I was young, we wrote this definition with even more mathematical formalism:

$$\lim_{x \to a} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \ni 0 < \left| x - a \right| < \delta \Rightarrow \left| f(x) - L \right| < \varepsilon.$$

In this highly symbolic form, the upside down capital “A” means “for every,” and the backwards “E” means “there exists.”

$$\lim_{x \to a} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \ni 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

So what does this actually mean?

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It means that if we specify any positive number *epsilon*, then we can always find a sufficiently small number *delta* such that if the distance between x and a is less than *delta* but greater than zero, then the distance between f(x) and L will be less than *epsilon*. If this works no matter how small we make *epsilon*, then L is the *limit* of our function as x approaches a.

Why do we require the distance between x and a to be greater than zero?

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Recall that the whole purpose of *limits* is to let us more rigorously define things such as the derivative.

In modern notation we write:

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

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If we let *x* equal *a* in this expression, then we are dividing by zero, and the universe as we know it ceases to exist.

Thus, in the concept of the *limit* as x goes to a, we always want to consider what happens as x gets close to a while remaining different from a.

This practice also works well in those instances where a function may not even be defined at *a*, or in cases where the function value at *a* may be different from the *limit* value.

$$\lim_{x \to 1} \frac{x-1}{x-1}$$

$$f(x) = \begin{cases} 2 & if \ x \neq 0 \\ -2 & if \ x = 0 \end{cases}$$

KEY THEOREMS ABOUT LIMITS

$$\textit{Suppose} \lim_{x\to a} f(x) = L \textit{ and } \lim_{x\to a} g(x) = M. \textit{ Then,}$$

1. $\lim_{x\to a}\left(f(x)+g(x)\right) = L + M$
2. $\lim_{x\to a}\left(f(x)-g(x)\right) = L - M$
3. $\lim_{x\to a}\left(f(x)\cdot g(x)\right) = L\cdot M$
4. $\lim_{x\to a}\left(f(x)/g(x)\right) = L/M$, *provided* $M \neq 0$
5. $\lim_{x\to a}\left(c\cdot f(x)\right) = c\cdot M$, *where* c *is a fixed value*

CONTINUITY

A function *f(x)* is continuous at *a* if:

$$\lim_{x \to a} f(x) = f(a)$$

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This is sometimes written as:

1. $f(a)$ *is defined*.
2. $\lim_{x \to a} f(x)$ *exists*.
3. $\lim_{x \to a} f(x) = f(a)$.

KEY THEOREMS ABOUT CONTINUITY

If *f* and *g* are continuous at *a* and if *c* is a constant, then the following functions are also continuous at *a*.

1. $f + g$
2. $f - g$
3. $f \cdot g$
4. $f\,/\,g,\ if\ \ g(a) \neq 0$
5. $c \cdot f$

LIMITS FOR FUNCTIONS OF TWO VARIABLES

Definition: The disk D with center *(a,b)* and radius *r* is defined as:

$$D = \left\{ (x, y) : \sqrt{\left(x - a\right)^2 + \left(y - b\right)^2} \leq r \right\}$$

LIMITS FOR FUNCTIONS OF TWO VARIABLES

Let *f* be a function of two variables defined on a disk *D* with center *(a,b)*. Then,

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \text{ if and only if for every } \varepsilon > 0 \text{ there exists a } \delta > 0$$

$$\text{such that if } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta, \text{ then } \left| f(x,y) - L \right| < \varepsilon.$$

The function *f* is continuous at *(a,b)* if and only if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

EXAMPLES:

$$\lim_{(x,y)\to(o,o)} \frac{x^2 - y^2}{x^2 + y^2} = ?$$

Approach *(0,0)* first along the *x*-axis and then the *y*-axis.

EXAMPLES:

$$\lim_{(x,y)\to(o,o)} \frac{x \cdot y}{x^2 + y^2} = ?$$

Approach *(0,0)* first along the *x*-axis, then along the *y*-axis, and finally along the line *y*=*x*.

EXAMPLES:

$$\lim_{(x,y)\to(o,o)} \frac{x^2 \cdot y}{x^4 + y^2} = ?$$

Approach *(0,0)* first along the *x*-axis, then along the *y*-axis, then along any line *y=mx,* and finally along the parabola $y=x^2$.

Prove: $$\lim_{(x,y)\to(o,o)} \frac{3x^2y}{x^2+y^2} = 0$$

Proof: *Let* $\varepsilon > 0$ *and let* $\delta = \frac{\varepsilon}{3}$. *Then,*

$$0 < \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2+y^2} < \delta = \frac{\varepsilon}{3} \Rightarrow$$

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| = \left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2} \le 3|y|$$

$$= 3\sqrt{y^2} \le 3\sqrt{x^2+y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

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Proof: $Let\, \varepsilon > 0\, and\, let\, \delta = \frac{\varepsilon}{3}.\, Then,$

$$0<\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}<\delta=\frac{\varepsilon}{3}\Rightarrow$$

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$$=3\sqrt{y^2}\le 3\sqrt{x^2+y^2}<3\delta=3\left(\frac{\varepsilon}{3}\right)=\varepsilon.$$

Therefore, $$\lim_{(x,y)\to(o,o)} \frac{3x^2y}{x^2+y^2}=0$$

Is *f(x,y)* continuous at *(2,3)*?

$$f(x, y) = x^2 + y^2$$

Is *f(x,y)* continuous at *(0,0)*?

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2} & if\ (x,y)\neq(0,0)\\ 0 & if\ (x,y)=(0,0)\end{cases}$$

Is *f(x,y)* continuous at *(0,0)*?

$$f(x,y)=\begin{cases} x^2+y^2 & if\ (x,y)\neq(0,0) \\ 1 & if\ (x,y)=(0,0) \end{cases}$$