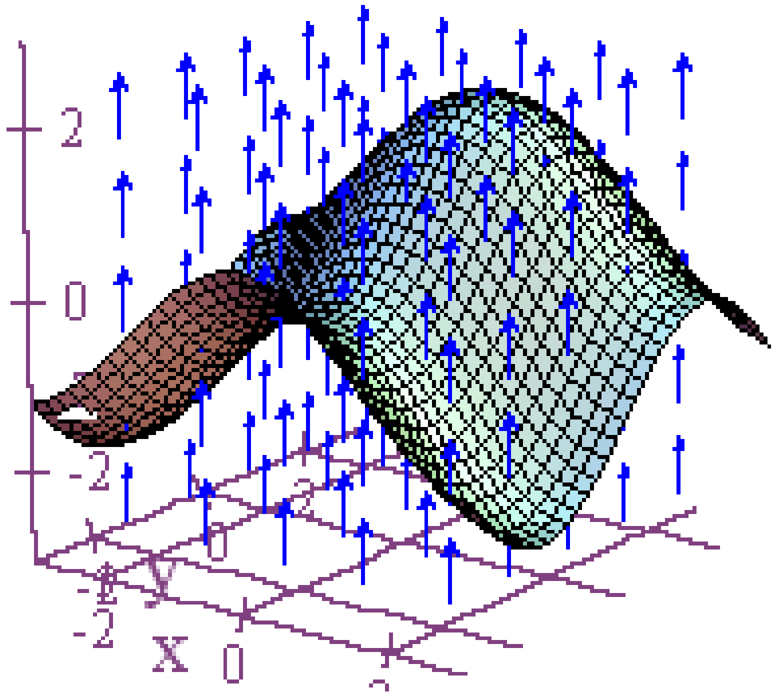
FLUX WITHOUT DIVERGENCE



If we are trying to find the flux through a surface that doesn't enclose a volume, then we can't use the Divergence Theorem. However, we can still try to compute the flux integral directly.

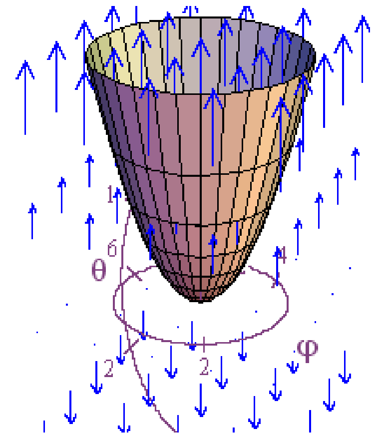
$$\text{Flux} = \iint_S \vec{F} \bullet N \, dS$$

To do so, we'll usually set up one of the following integrals. Also, we may sometimes need to adjust a sign depending upon whether we want our unit normal vector to point upwards or downwards.

$$\text{Flux} = \iint_S \vec{F} \bullet N \, dS = \iint_R \left(\vec{F} \bullet \frac{\nabla g}{\left\| \nabla g \right\|} \right) \left\| \nabla g \right\| dy dx = \iint_R \vec{F} \bullet \nabla g \, dy dx$$

$$\text{Flux} = \iint_S \vec{F} \bullet N \, dS = \iint_R \left(\vec{F} \bullet \frac{\vec{r}_s \times \vec{r}_t}{\left\| \vec{r}_s \times \vec{r}_t \right\|} \right) \left\| \vec{r}_s \times \vec{r}_t \right\| ds dt = \iint_R \vec{F} \bullet \left(\vec{r}_s \times \vec{r}_t \right) ds dt$$

Example 1:



$$\vec{F} = z\hat{k}$$

$$z = x^2 + y^2$$

$R =$ unit circle

$N =$ upward unit normal

$$g = -x^2 - y^2 + z$$

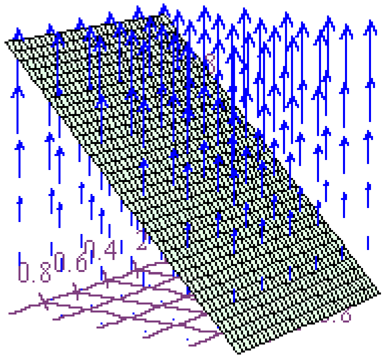
$$\nabla g = -2x\hat{i} - 2y\hat{j} + \hat{k}$$

$$N\,dS = \frac{\nabla g}{\|\nabla g\|} \cdot \|\nabla g\|\,dydx = \nabla g\,dydx$$

$$\iint_S \vec{F} \bullet N\,dS = \iint_R \vec{F} \bullet \nabla g\,dydx = \iint_R \left(z\hat{k} \bullet (-2x\hat{i} - 2y\hat{j} + \hat{k}) \right) dydx = \iint_R z\,dydx$$

$$= \iint_R (x^2 + y^2)\,dydx = \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = \int_0^{2\pi} \left.\frac{r^4}{4}\right|_0^1 d\theta = \int_0^{2\pi} \frac{1}{4}\,d\theta = \left.\frac{\theta}{4}\right|_0^{2\pi} = \frac{\pi}{2}$$

Example 2:



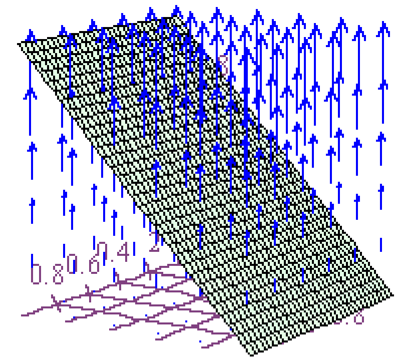
$$\vec{F} = z\hat{k}$$
$$z = -y + 1$$
$$R = 0 \le x \le 1, 0 \le y \le 1$$
$N =$ upward unit normal

$$g = y + z$$
$$\nabla g = \hat{j} + \hat{k}$$
$$N\,dS = \frac{\nabla g}{\|\nabla g\|} \cdot \|\nabla g\|\,dydx = \nabla g\,dydx$$

$$\iint_S \vec{F} \bullet N\,dS = \iint_R \vec{F} \bullet \nabla g\,dydx = \iint_R \left(z\hat{k} \bullet (\hat{j} + \hat{k})\right) dydx = \iint_R z\,dydx$$

$$= \int_0^1 \int_0^1 (-y+1)\,dydx = \int_0^1 \left(\frac{-y^2}{2} + y\right)\Bigg|_0^1 dx = \int_0^1 \frac{1}{2}\,dx = \frac{x}{2}\Bigg|_0^1 = \frac{1}{2}$$

Example 3:



$$\vec{F} = z\hat{k}$$

$$\vec{u} = \hat{i}$$

$$\vec{v} = \hat{j} - \hat{k}$$

$$P = \langle 0,0,1 \rangle$$

$$\vec{r}(s,t) = P + s\vec{u} + t\vec{v}$$

$$= s\hat{i} + t\hat{j} + (1-t)\hat{k}$$

$$R = 0 \le \mathrm{s} \le 1, 0 \le \mathrm{t} \le 1$$

$N =$ upward unit normal

$$\vec{r}_s = \hat{i}$$

$$\vec{r}_t = \hat{j} - \hat{k}$$

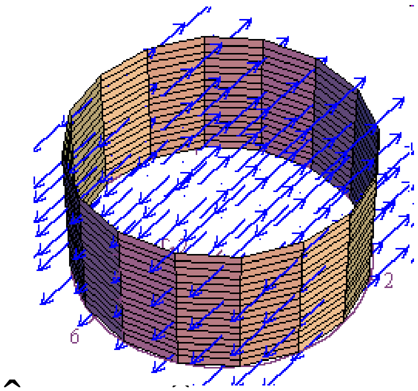
$$\vec{r}_s \times \vec{r}_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \hat{j} + \hat{k}$$

$$N\,dS = \frac{\vec{r}_s \times \vec{r}_t}{\left\|\vec{r}_s \times \vec{r}_t\right\|} \cdot \left\|\vec{r}_s \times \vec{r}_t\right\| dsdt = \left(\vec{r}_s \times \vec{r}_t\right) dsdt$$

$$\iint_S \vec{F} \bullet N\,dS = \iint_R \vec{F} \bullet \left(\vec{r}_s \times \vec{r}_t\right) dsdt = \iint_R \left(z\hat{k} \bullet (\hat{j} + \hat{k})\right) dydx = \iint_R z\,dsdt$$

$$= \int_0^1 \int_0^1 (1-t)\,dsdt = \int_0^1 \left(t - \frac{t^2}{2}\right)\Bigg|_0^1 dt = \frac{1}{2}$$

Example 4:



$$\vec{F} = \frac{x}{\sqrt{x^2+y^2}}\hat{i} = \cos\theta\hat{i}$$

$$\vec{r}(\theta,z) = \cos\theta\hat{i} + \sin\theta\hat{j} + z\hat{k}$$

$$R = 0 \le \theta \le 2\pi, 0 \le \mathrm{z} \le 1$$

$N =$ outward unit normal

$$\vec{r}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

$$\vec{r}_z = \hat{k}$$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

$$N\,dS = \frac{\vec{r}_\theta \times \vec{r}_z}{\left\|\vec{r}_\theta \times \vec{r}_z\right\|} \cdot \left\|\vec{r}_\theta \times \vec{r}_z\right\| dzd\theta = \left(\vec{r}_\theta \times \vec{r}_z\right) dzd\theta$$

$$\iint_S \vec{F} \bullet N\,dS = \iint_R \vec{F} \bullet \left(\vec{r}_\theta \times \vec{r}_z\right) dzd\theta = \iint_R \left(\cos\theta\hat{i} \bullet (\cos\theta\hat{i} + \sin\theta\hat{j})\right) dzd\theta$$

$$= \iint_R \cos^2\theta\,dzd\theta = \int_0^{2\pi}\int_0^1 \cos^2\theta\,dzd\theta = \int_0^{2\pi} \left(z\cos^2\theta\right)\Big|_0^1 d\theta = \int_0^{2\pi} \cos^2\theta\,d\theta$$

$$= \int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta = \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Bigg|_0^{2\pi} = \pi$$