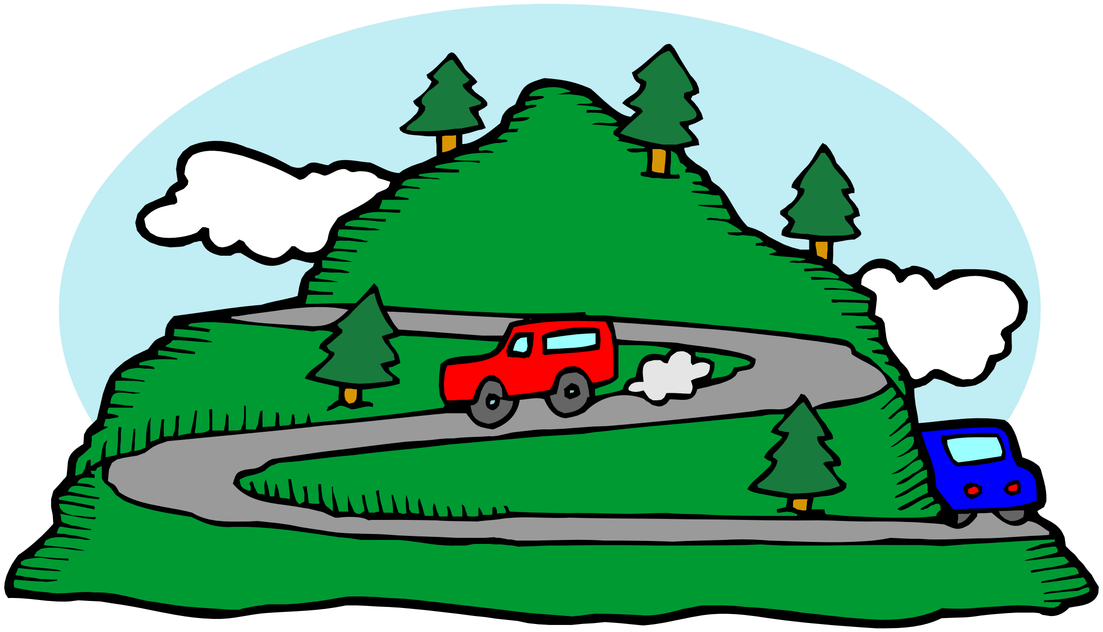
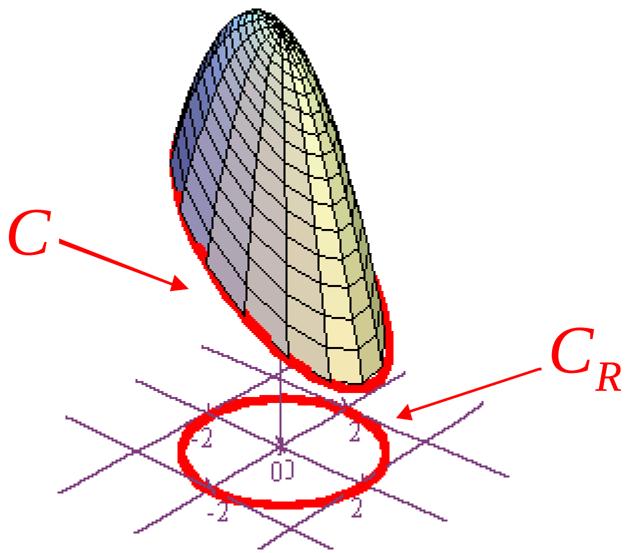
STOKES' THEOREM IN HIGHER DIMENSIONS



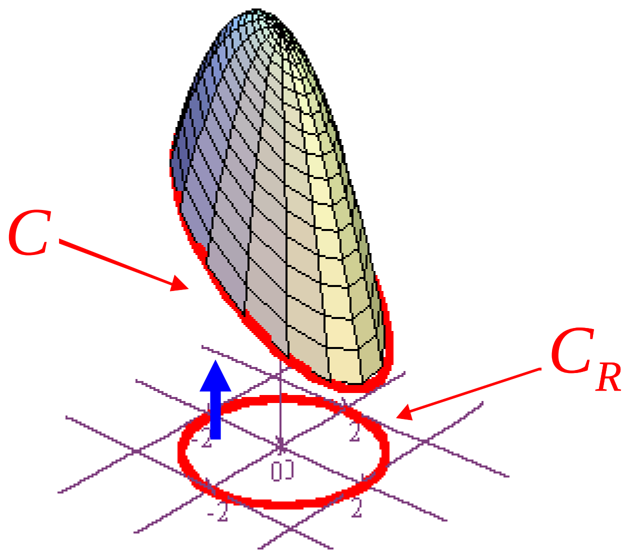
If we apply Stokes' Theorem to a vector field $F = \langle P, Q \rangle$ and a plane curve C_R that is oriented counterclockwise and that bounds a region R, then we get the following formula (the same as Green's Theorem):

$$\int_{C_R} F \bullet dr = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R curl\, F \bullet \hat{k}\, dA = \iint_R (\nabla \times F) \bullet \hat{k}\, dA$$

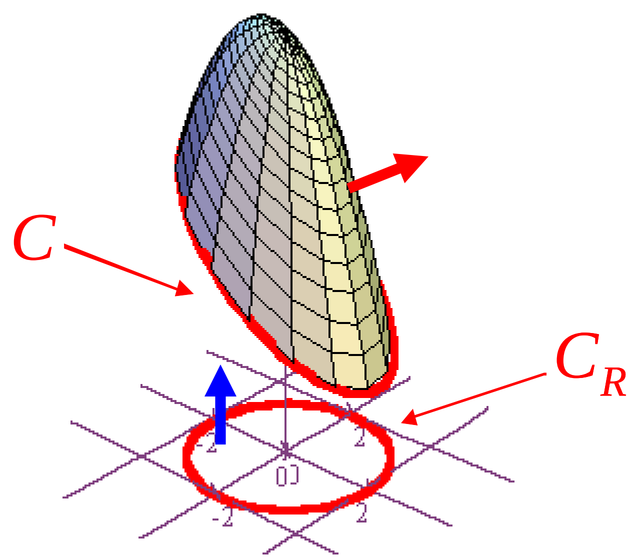


Notice in this formula that we are integrating the dot product of the curl of *F* with an upward pointing unit normal vector *k*.

$$\int_{C_R} F \bullet dr = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R curl\, F \bullet \hat{k}\, dA = \iint_R \left(\nabla \times F \right) \bullet \hat{k}\, dA$$

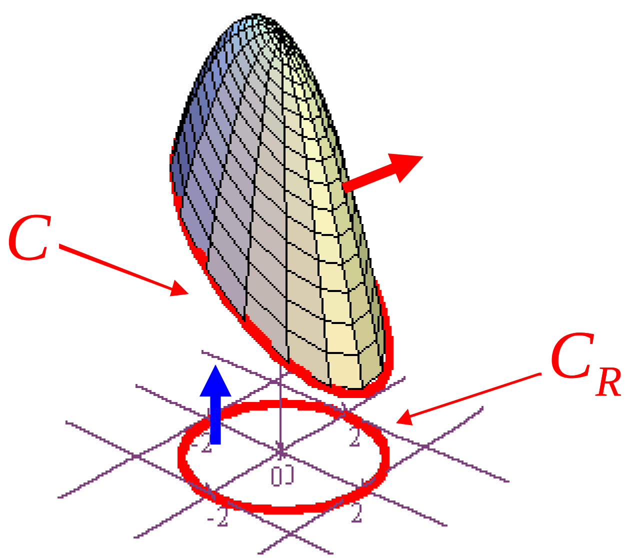


Thus, it should come as no surprise that if we integrate around a counterclockwise oriented curve *C* that bounds a surface *S*, then our formula will involve both an upward pointing unit normal and a surface integral.



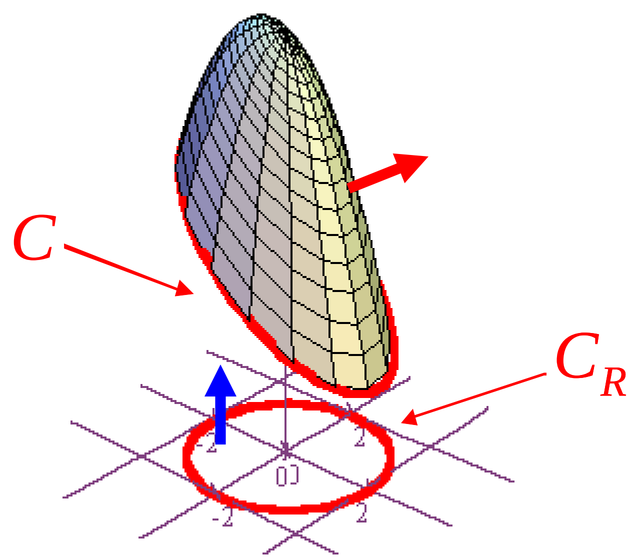
Let's suppose that our surface is the graph of *z=f(x,y).* Then consider this as a level surface for the function *g(x,y,z)=z-f(x,y)*, and define an upward pointing unit normal as follows:

$$N = \frac{\nabla g}{\|\nabla g\|} = \frac{-f_x\,\hat{i} - f_y\,\hat{j} + \hat{k}}{\sqrt{f_x^2 + f_y^2 + 1}}$$



Notice that our unit normal will point upward because the *k* component is positive.

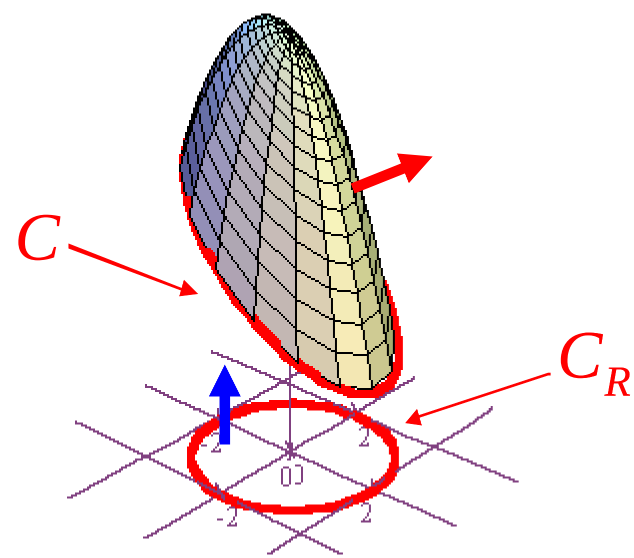
$$N = \frac{\nabla g}{\left\| \nabla g \right\|} = \frac{-f_x \hat{i} - f_y \hat{j} + \hat{k}}{\sqrt{f_x^2 + f_y^2 + 1}}$$



We can now write out the higher dimensional version of Stokes' Theorem.

$$\int_C F \bullet dr = \iint_S (\nabla \times F) \bullet N \, dS = \iint_S (\nabla \times F) \bullet \frac{\nabla g}{\|\nabla g\|} dS$$

$$= \iint_R (\nabla \times F) \bullet \frac{\nabla g}{\sqrt{f_x^2 + f_y^2 + 1}} \sqrt{f_x^2 + f_y^2 + 1} \, dA = \iint_R (\nabla \times F) \bullet \nabla g \, dA$$



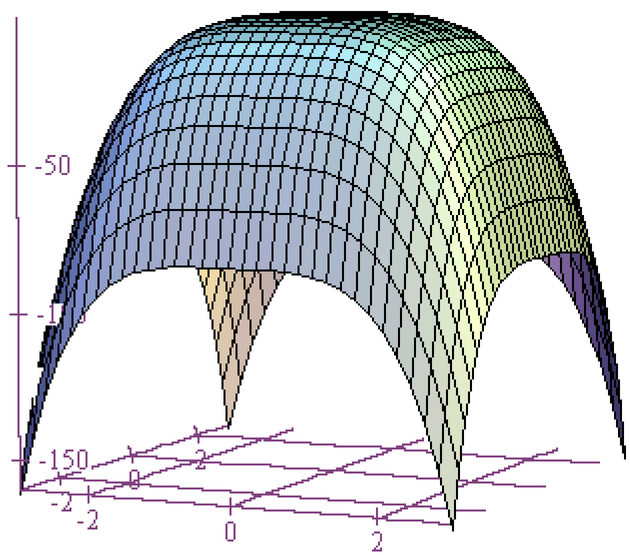
Now let's do a problem!

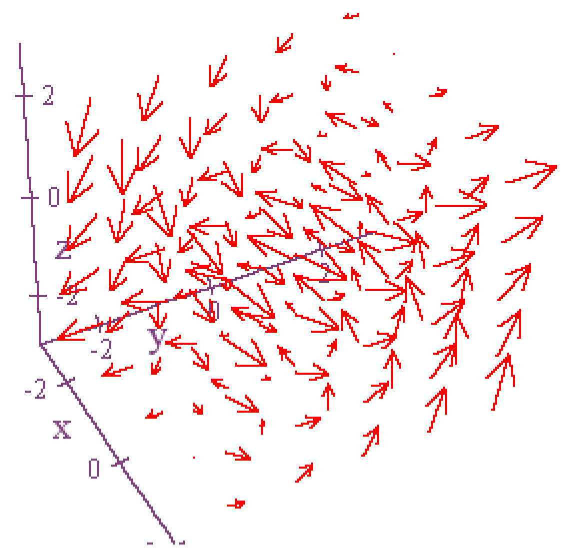
$S: z = -x^4 - y^4$

$R: 0 \le x \le 1,\ 0 \le y \le 1$

$F = z\hat{i} + x\hat{j} + y\hat{k}$

$$\int_C F \bullet dr = \iint_R (\nabla \times F) \bullet \nabla g \, dA$$





First, find the gradient of *g* and the curl of *F*.

$$S: z = -x^4 - y^4$$

$$R: 0 \le x \le 1,\ 0 \le y \le 1$$

$$F = z\,\hat{i} + x\,\hat{j} + y\,\hat{k}$$

$$\int_C F \bullet dr = \iint_R \left(\nabla \times F\right) \bullet \nabla g\, dA$$

$$g = x^4 + y^4 + z$$

$$\nabla g = 4x^3\,\hat{i} + 4y^3\,\hat{j} + \hat{k}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

And now, integrate!

$$S : z = -x^4 - y^4$$
$$R : 0 \le x \le 1,\ 0 \le y \le 1$$
$$F = z\hat{i} + x\hat{j} + y\hat{k}$$

$$\int_C F \bullet dr = \iint_R (\nabla \times F) \bullet \nabla g \, dA$$

$$g = x^4 + y^4 + z$$
$$\nabla g = 4x^3\hat{i} + 4y^3\hat{j} + \hat{k}$$
$$\nabla \times F = \hat{i} + \hat{j} + \hat{k}$$

$$\int_C \vec{F} \bullet d\vec{r} = \iint_R \left[(\nabla \times F) \bullet \nabla g \right] dA = \int_0^1 \int_0^1 \left[\left(\hat{i} + \hat{j} + \hat{k} \right) \bullet \left(4x^3\hat{i} + 4y^3\hat{j} + \hat{k} \right) \right] dy dx$$

$$= \int_0^1 \int_0^1 \left(4x^3 + 4y^3 + 1 \right) dy dx = \int_0^1 4x^3 y + y^4 + y \Big|_0^1 dx = \int_0^1 (4x^3 + 2)\, dx$$

$$= x^4 + 2x \Big|_0^1 = 1 + 2 = 3$$