Independence of Path

DEFINITION: A vector field $\vec{F} = P\hat{i} + Q\hat{j}$ is conservative if $\vec{F} = \nabla f$ for some function $z = f(x, y)$. In this case, f is called the potential function for $\vec{F}$.

DEFINITION: The line integral $\int_C \vec{F} \bullet d\vec{r}$ is independent of path if $\int_{C_1} \vec{F} \bullet d\vec{r} = \int_{C_2} \vec{F} \bullet d\vec{r}$ for any two paths C_1 and C_2 that have the same initial and terminal points.

THEOREM: The line integral $\int_C \vec{F} \bullet d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F} \bullet d\vec{r} = 0$ for every closed path C in D.

The Fundamental Theorem of Line Integrals: Let C be a smooth curve with a smooth parametrization $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ for $a \leq t \leq b$, and let $z = f(x, y)$ be a function whose gradient ∇f is continuous on C. Then $\int_C \nabla f \bullet d\vec{r} = f\left(x(b), y(b)\right) - f\left(x(a), y(a)\right)$.

PROOF: $\int_C \nabla f \bullet d\vec{r} = \int_a^b \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \right) \bullet \frac{d\vec{r}}{dt} dt$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \right) \bullet \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \right) dt = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right) dt$$

$$= \int_a^b \frac{df}{dt} dt = f\left(x(b), y(b)\right) - f\left(x(a), y(a)\right).$$

DEFINITION: A region R is connected if any two points in R can be joined by a path C that lies in R. A connected region R is simply connected if it contains no holes.

THEOREM: Let $\vec{F} = P\hat{i} + Q\hat{j}$ be a vector field defined on an open simply connected region $R,$ let C be a smooth curve in R, and suppose P and Q have continuous first order derivatives in R. Then the following are equivalent.

1. $\vec{F} = P\hat{i} + Q\hat{j}$ is conservative.
2. $\vec{F} = \nabla f$ for some function $z = fx, y)$.
3. $\int_C \vec{F} \bullet d\vec{r}$ is independent of path.
4. $\int_C \vec{F} \bullet d\vec{r} = 0$ for every closed curve C in R.
5. $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.
6. $curl\, \vec{F} = \nabla \times \vec{F} = \vec{0}$

Example 1: $\vec{F} = 2x\hat{i} + 2y\hat{j}$ and C is any path from $(1,1)$ to $(2,2)$.

Find $\int_C \vec{F} \bullet d\vec{r}$.

In this case, if $f(x,y) = x^2 + y^2$, then $\nabla f = \vec{F}$. Thus, the integral is independent of path and $\int_C \vec{F} \bullet d\vec{r} = f(2,2) - f(1,1) = 8 - 2 = 6$.

Example 2: If $\vec{F} = (3+2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$, show that $\int_C \vec{F} \bullet d\vec{r}$ is independent of path.

Let $P = 3+2xy$ and $Q = x^2 - 3y^2$. Then $\dfrac{\partial P}{\partial y} = 2x = \dfrac{\partial Q}{\partial x}$.

Therefore, $F = \nabla f$ for some function $z = f(x, y)$, and $\int_C \vec{F} \bullet d\vec{r}$ is independent of path.

Example 3: If $\vec{F} = (3 + 2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$, find a potential function $z = f(x, y)$.

Let $P = 3 + 2xy$ and $Q = x^2 - 3y^2$. Then $\int P\, dx = 3x + x^2 y + g(y)$.

Differentiate this result with respect to y and you get $x^2 + g'(y)$.

Comparing this result with $Q = x^2 - 3y^2$, we see that we want $g'(y) = -3y^2$. An antiderivative of this with respect to y is $-y^3$.

Hence, it suffices to let $f(x, y) = 3x + x^2 y - y^3$.

Example 4: If $\vec{F} = (3+2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$, find $\int_C \vec{F} \bullet d\vec{r}$

where the curve C is defined by $\vec{r}(t) = e^t \sin(t)\hat{i} + e^t \cos(t)\hat{j}$

where $0 \le t \le \pi$,

On this curve, $x = e^t \sin(t)$ and $y = e^t \cos(t)$. Also, the integral is independent of path, and a potential function for $\vec{F}$ is

$f(x, y) = 3x + x^2 y - y^3$. Hence, $\int_C \vec{F} \bullet d\vec{r} = f\left(x(\pi), y(\pi)\right) - f\left(x(0), y(0)\right)$

$$= f(e^\pi \sin\pi, e^\pi \cos\pi) - f(e^0 \sin(0), e^0 \cos(0))$$

$$= f(0, -e^\pi) - f(0,1)$$

$$= (3 \cdot 0 + 0^2(-e^\pi) - (-e^\pi)^3) - (3 \cdot 0 + 0^2(1) - 1^3)$$

$$= e^{3\pi} + 1.$$

Example 5: If $\vec{F} = y^2\hat{i} + (2xy+z)\hat{j} + (y+3z^2)\hat{k}$, find $\int_C \vec{F} \bullet d\vec{r}$

where C is any curve from $(0,0,0)$ to $(1,2,3)$.

To show that we have a gradient field, we have to verify that

$(y^2)_y = 2y = (2xy+z)_x, (y^2)_z = 0 = (y+3z^2)_x,$ and $(2xy+z)_z = 1 = (y+3z^2)_z$.

(Why?) Now that you know you have a gradient field, the steps below show you how to find a potential function.

$$\int y^2\,dx == f(x,y,z) = xy^2 + g(y,z)$$

$$2xy + z = f_y = 2xy + g_y \Rightarrow f(x,y,z) = xy^2 + yz + h(z)$$

$$y + 3z^2 = f_z = y + h_z \Rightarrow f(x,y,z) = xy^2 + yz + z^3$$

Example 5: If $\vec{F} = y^2\hat{i} + (2xy + z)\hat{j} + (y + 3z^2)\hat{k}$, find $\int_C \vec{F} \bullet d\vec{r}$ where C is any curve from $(0,0,0)$ to $(1,2,3)$.

Now that we know that $f(x,y,z) = xy^2 + yz + z^3$, it follows that

$$\int_C \vec{F} \bullet d\vec{r} = f(1,2,3) - f(0,0,0) = 4 + 6 + 27 - 0 = 37.$$