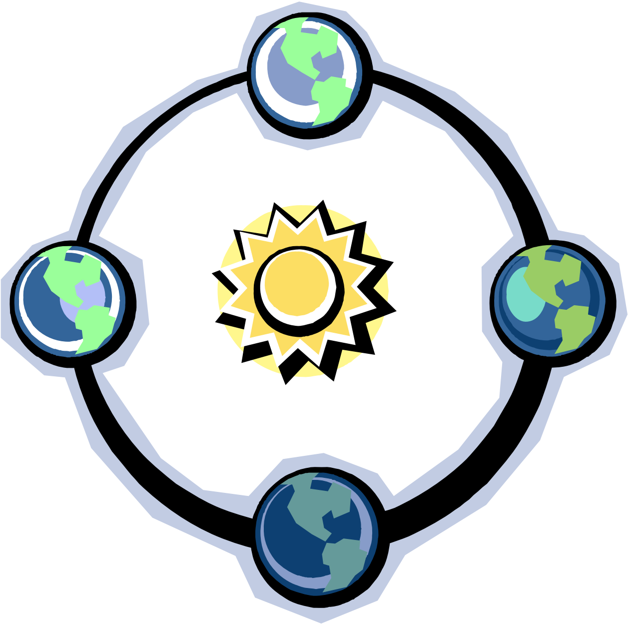
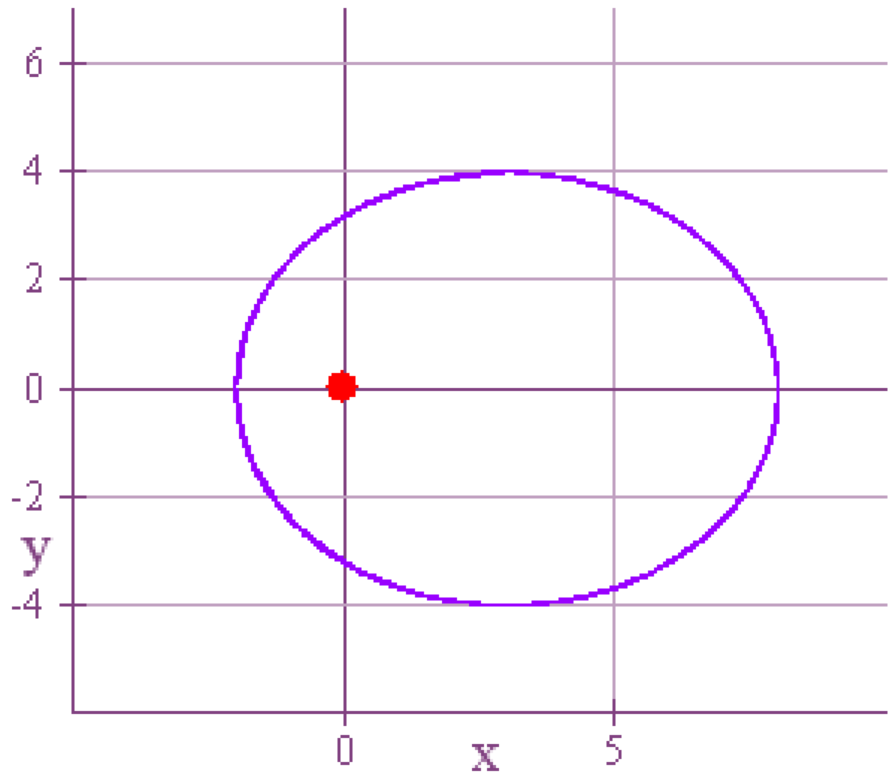
KEPLER'S SECOND LAW OF PLANETARY MOTION

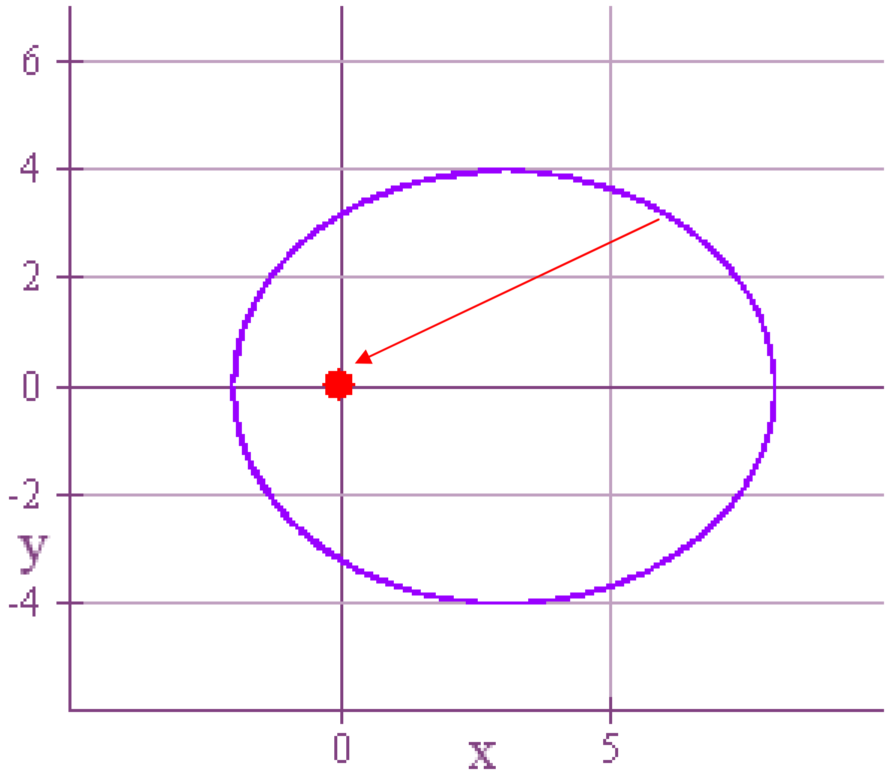
Kepler’s First Law says that the orbit of a planet is an ellipse with the sun at one focus.



Thus, suppose that this path is parametrized by *r(t)* with the sun placed at the origin.

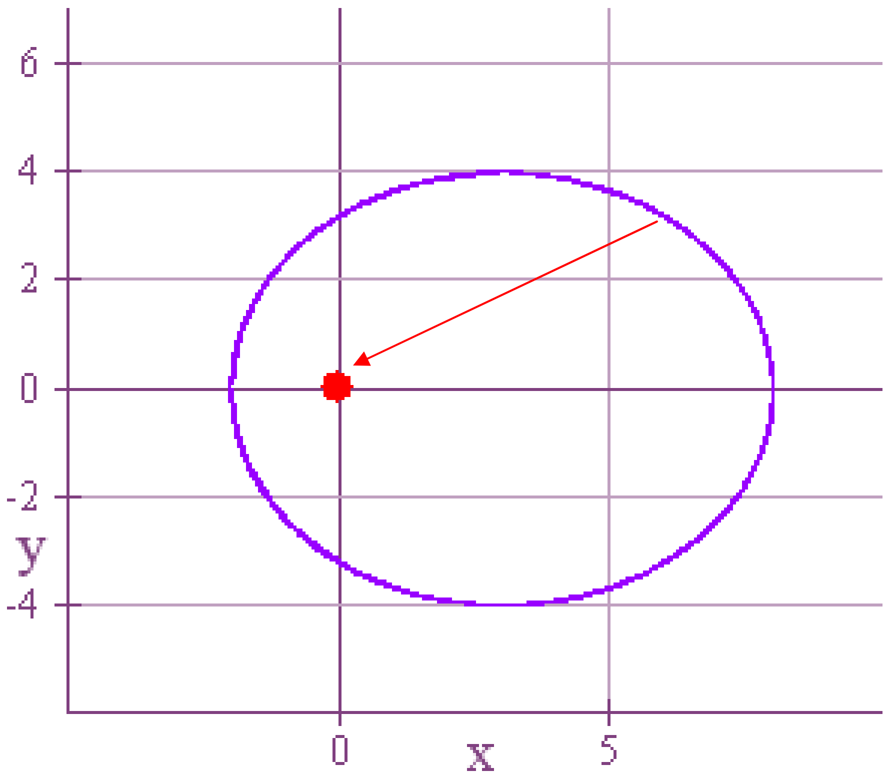


Then by Newton's Universal Law of Gravitation, the force of gravitational pull by the sun can be represented by the following vector.



$$F = -\frac{GMm}{\left\|\vec{r}(t)\right\|^2}\frac{\vec{r}(t)}{\left\|\vec{r}(t)\right\|} = -\frac{GMm}{\left\|\vec{r}(t)\right\|^3}\vec{r}(t)$$

On the other hand, Newton also showed that *Force = mass x acceleration*. Hence, …



$$F = m\vec{a} = m\vec{r}''(t)$$

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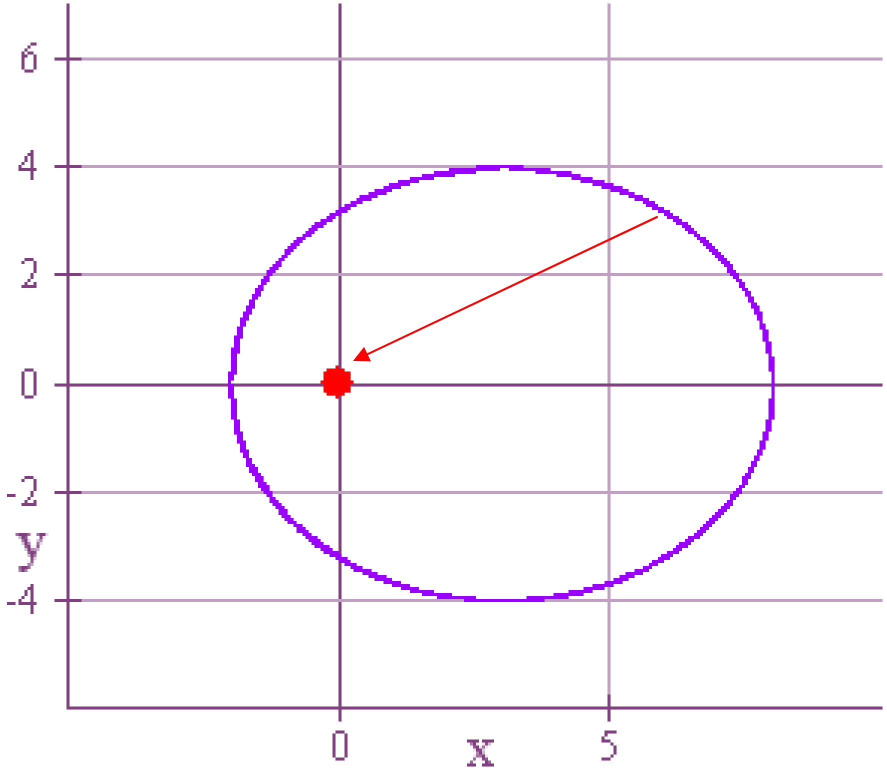
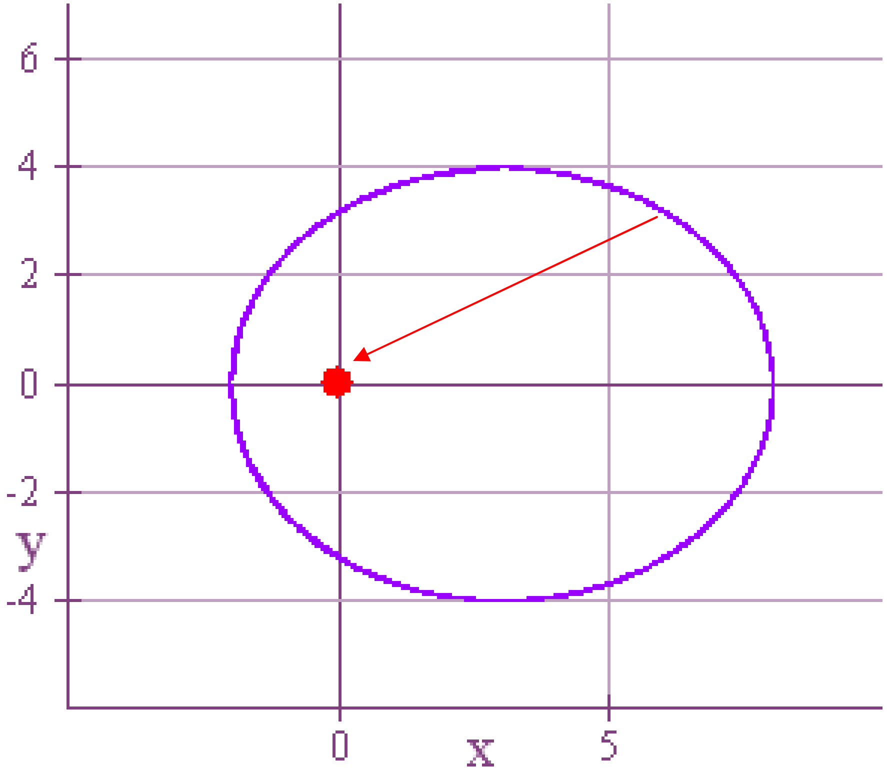
Therefore, …



$$m\vec{r}''(t) = -\frac{GMm}{\left\|\vec{r}(t)\right\|^3}\vec{r}(t) \Rightarrow \vec{r}''(t) = -\frac{GM}{\left\|\vec{r}(t)\right\|^3}\vec{r}(t)$$

Consequently, $\vec{r}''(t) \parallel \vec{r}(t)$.



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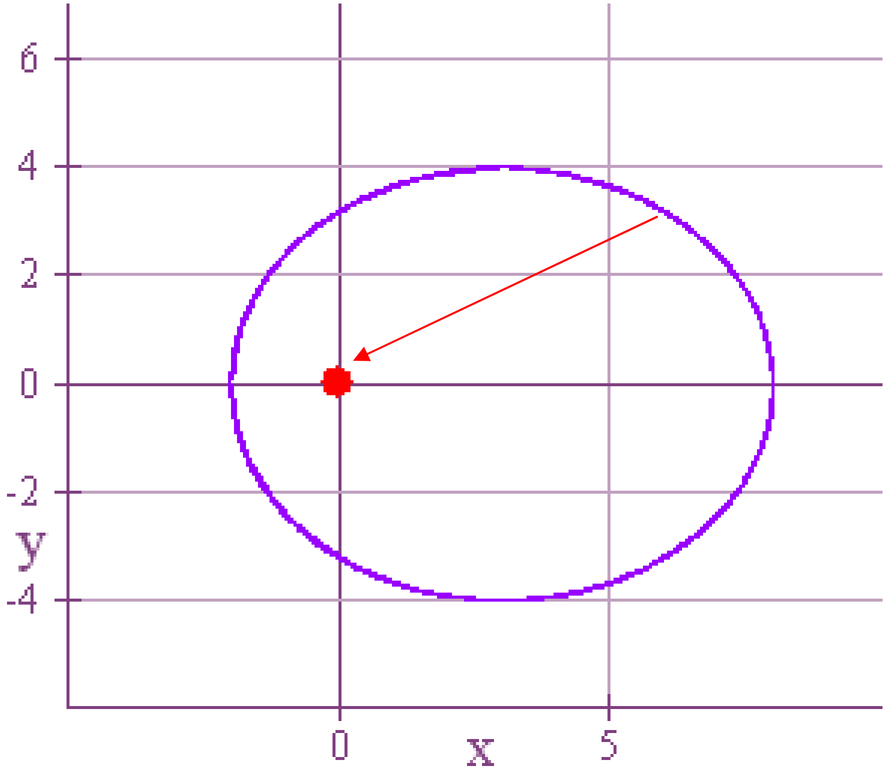
We'll use this fact shortly.



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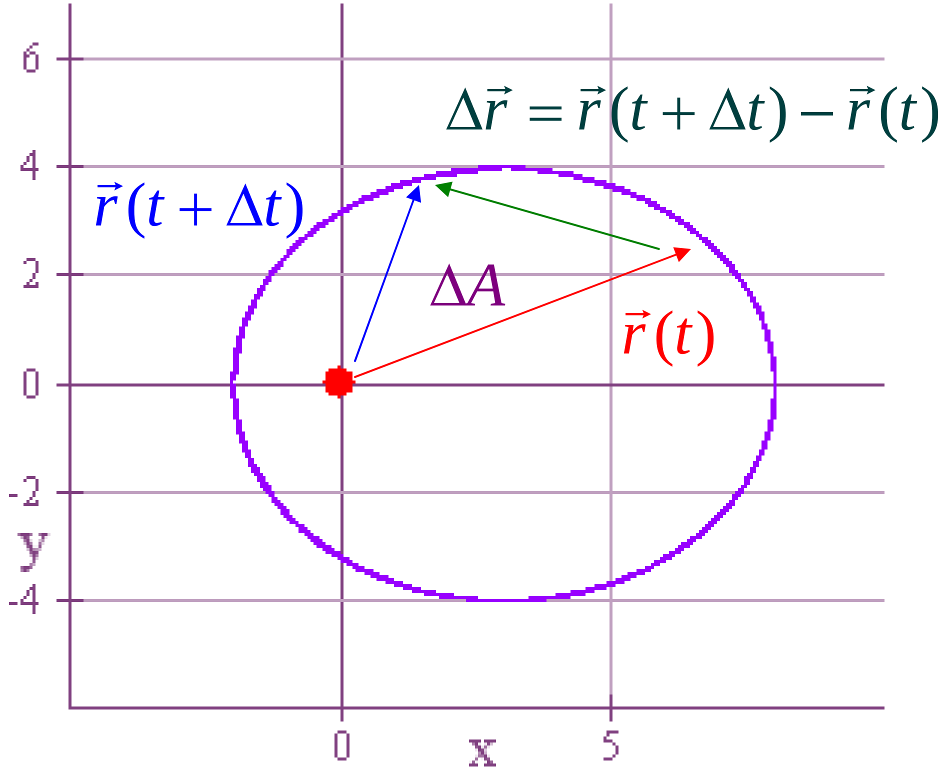
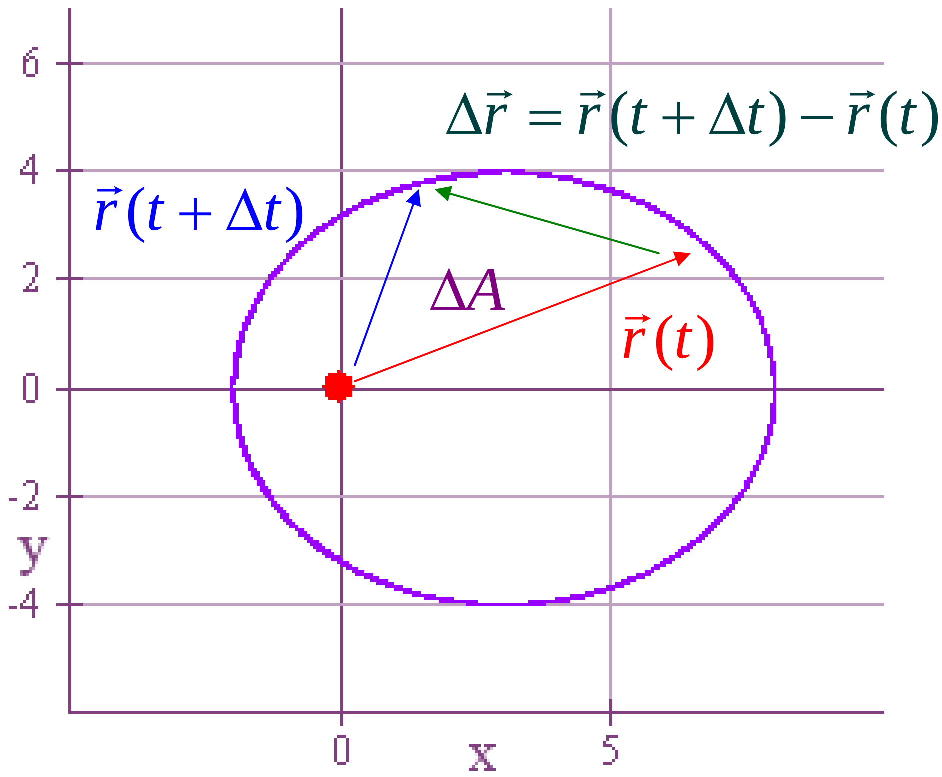
Another way to say this is that the rate of change of area with respect to time is constant.

Thus, let’s consider the following diagram.

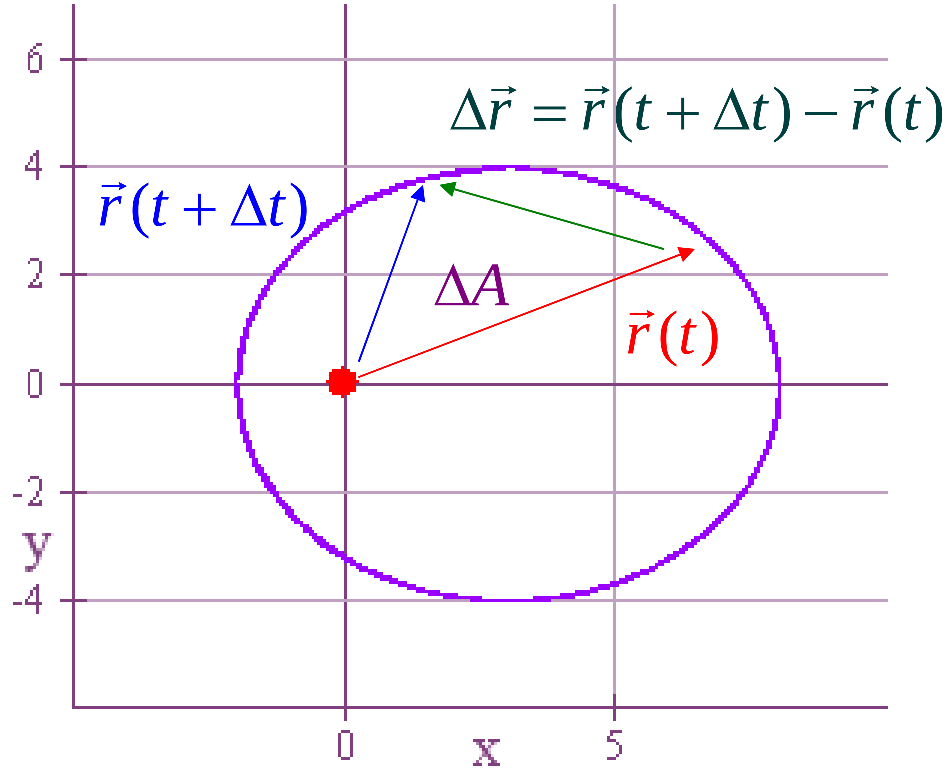


For small changes in *t*, the change in area is approximately the area of the triangle below.



Using the cross product, we get the following.

$$\Delta A \approx \frac{1}{2}\left\|\vec{r}(t)\times\Delta\vec{r}\right\|$$



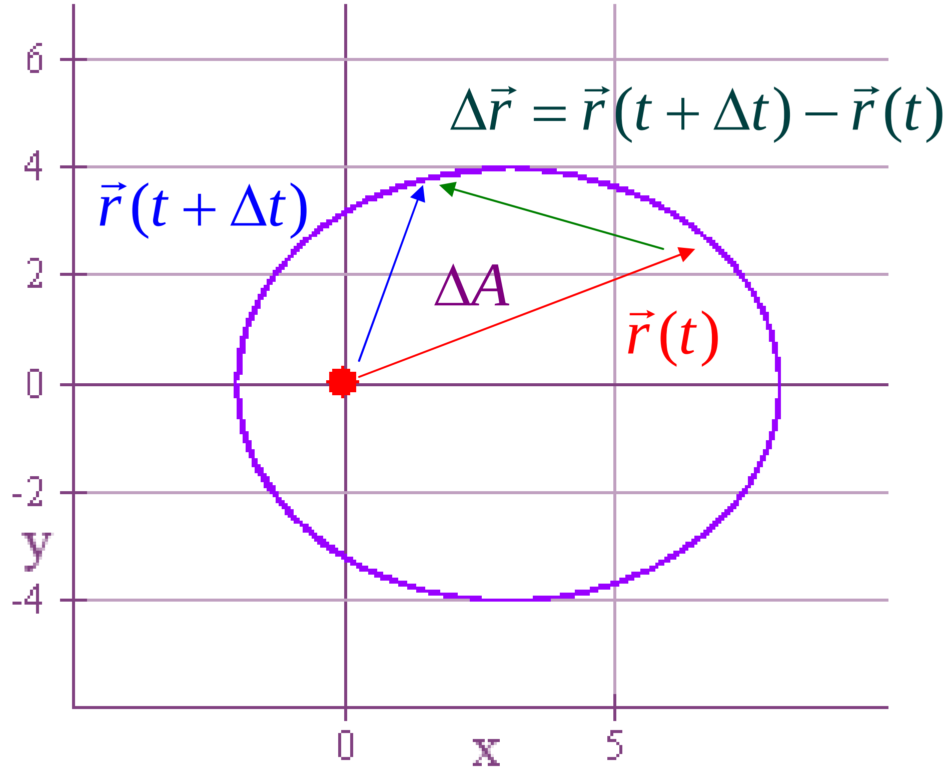
And now, we have consequences.

$$\Delta A \approx \frac{1}{2}\left\| \vec{r}(t) \times \Delta\vec{r} \right\|$$

$$\Rightarrow \frac{\Delta A}{\Delta t} \approx \frac{1}{2} \cdot \frac{1}{\Delta t}\left\| \vec{r}(t) \times \Delta\vec{r} \right\|$$

$$= \frac{1}{2}\left\| \vec{r}(t) \times \frac{\Delta\vec{r}}{\Delta t} \right\|$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2}\left\| \vec{r}(t) \times \vec{r}\,'(t) \right\|$$



We'll show that this derivative is constant by showing that the derivative of *r x r'* is the zero vector.

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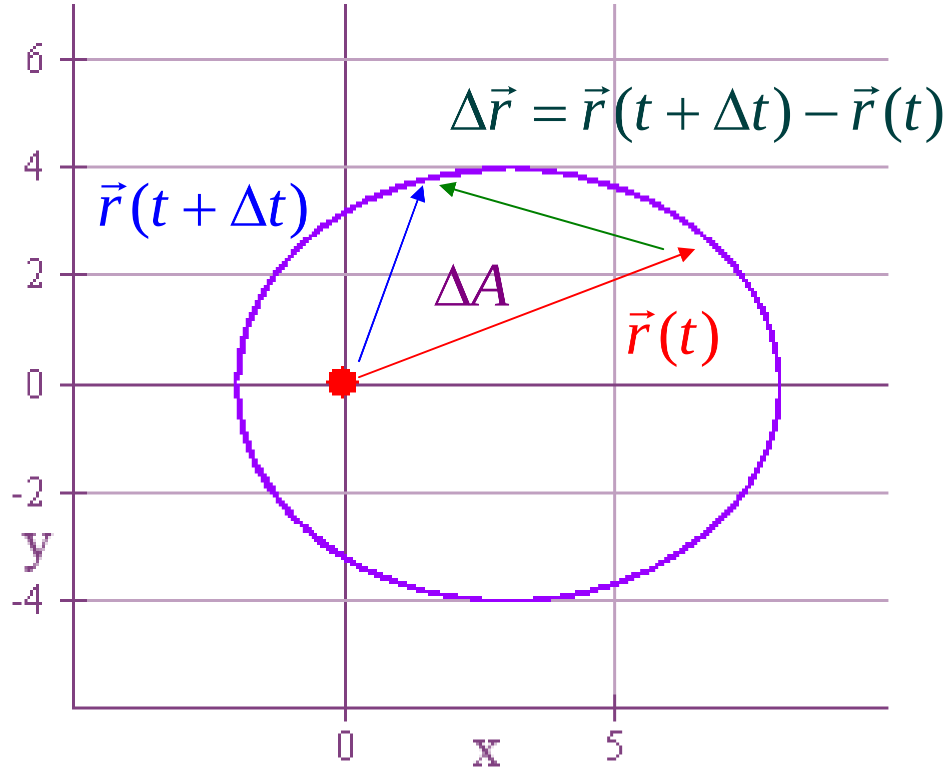

In the proof below, remember that we've already shown that *r* and *r*'' are parallel.

$$\frac{d\left(\vec{r}(t)\times\vec{r}'(t)\right)}{dt}=\vec{r}(t)\times\vec{r}''(t)+\vec{r}'(t)\times\vec{r}'(t)=\vec{0}+\vec{0}=\vec{0}.\ \square$$

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Therefore, $\vec{r}(t)\times\vec{r}\,'(t)$ is a constant vector

$$\Rightarrow \frac{dA}{dt}=\frac{1}{2}\left\|\vec{r}(t)\times\vec{r}\,'(t)\right\|=\text{constant}$$

$\Rightarrow$ Area changes at a constant rate

$\Rightarrow$ Equal areas are swept out by a position vector in equal amounts of time.

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