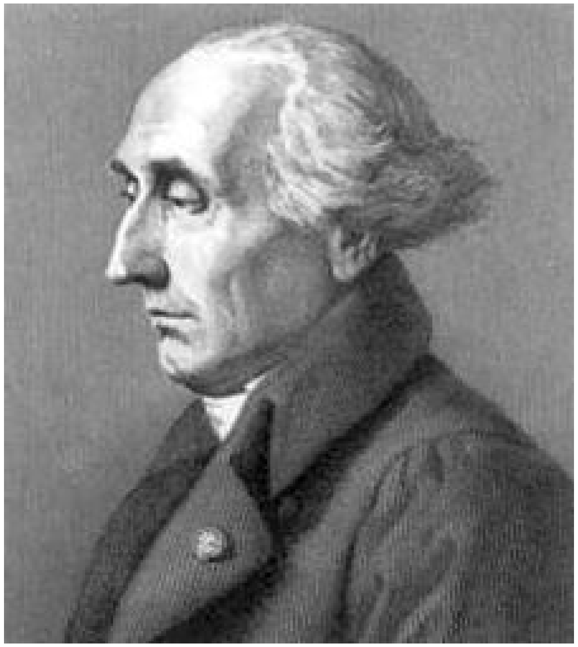
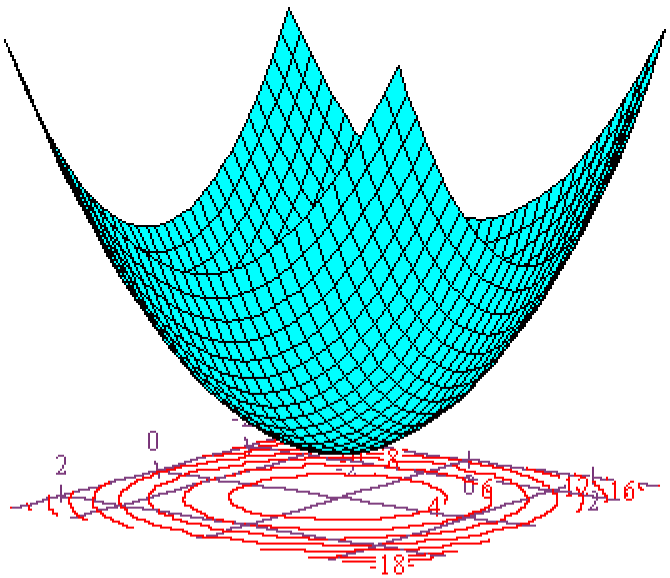
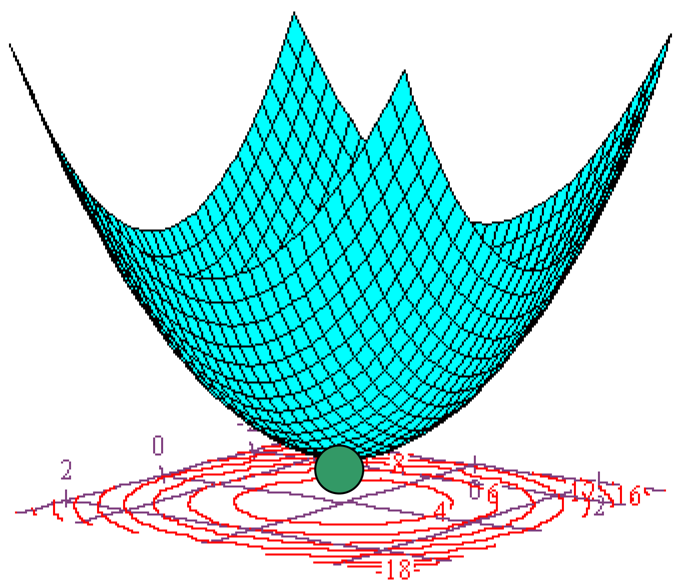
LAGRANGE MULTIPLIERS



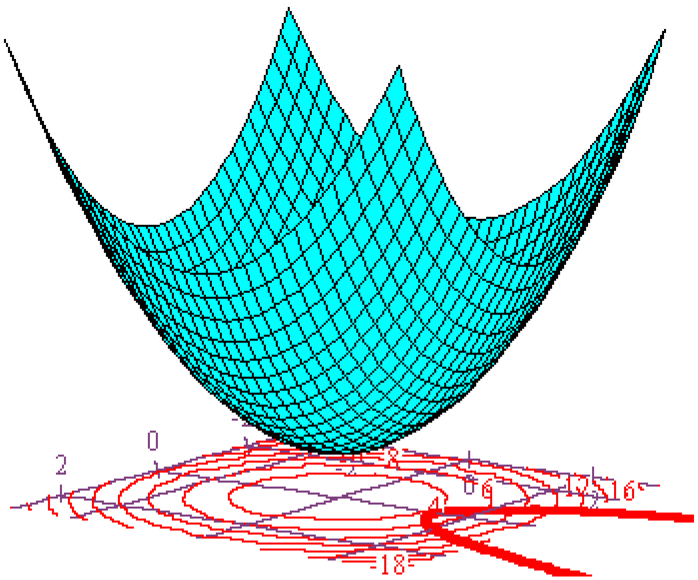
Let's start with a simple surface, $z=f(x,y)$.



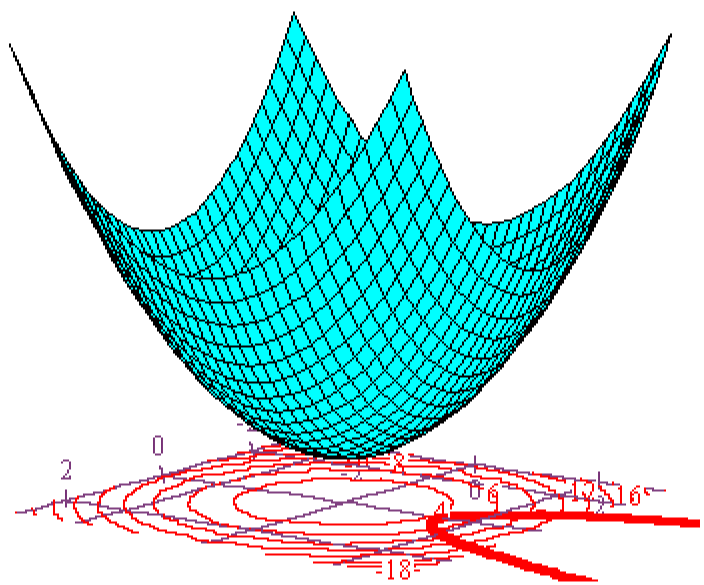
Clearly, this surface has a minimum point.



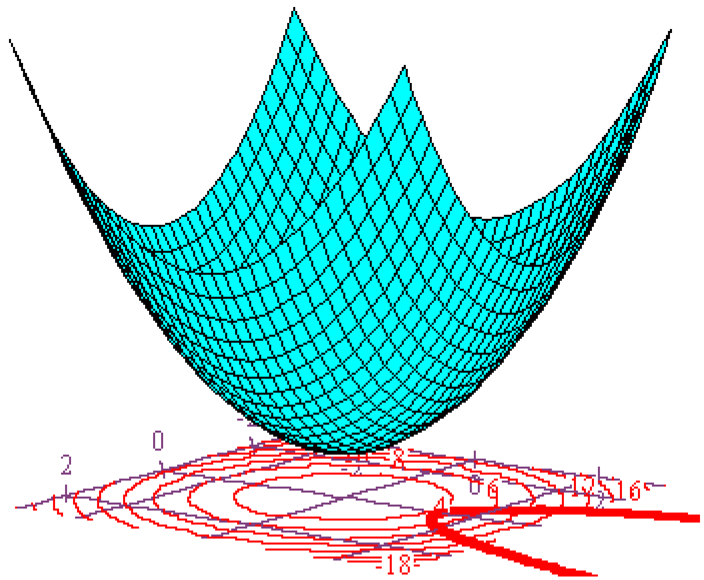
Now, down in the *xy*-plane, let's add a curve, *g(x,y)=c*.



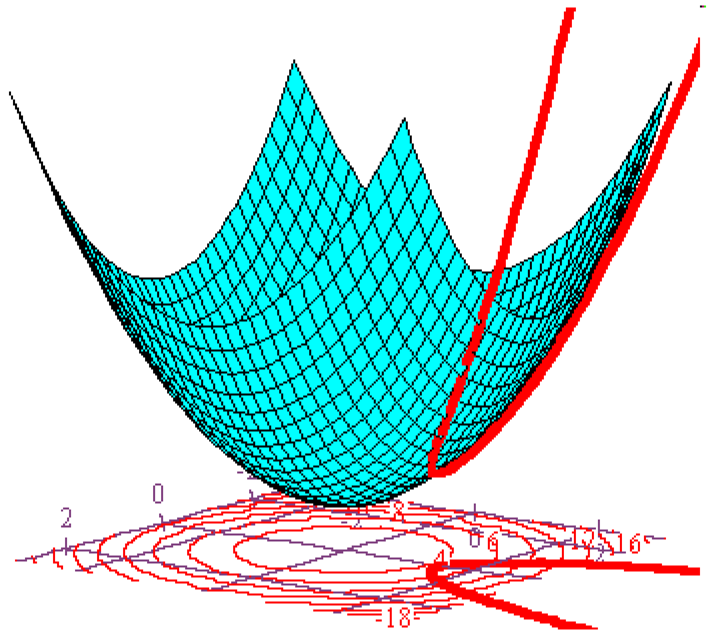
We can think of this curve as a level curve for a more general surface graph, *g=g(x,y)*.



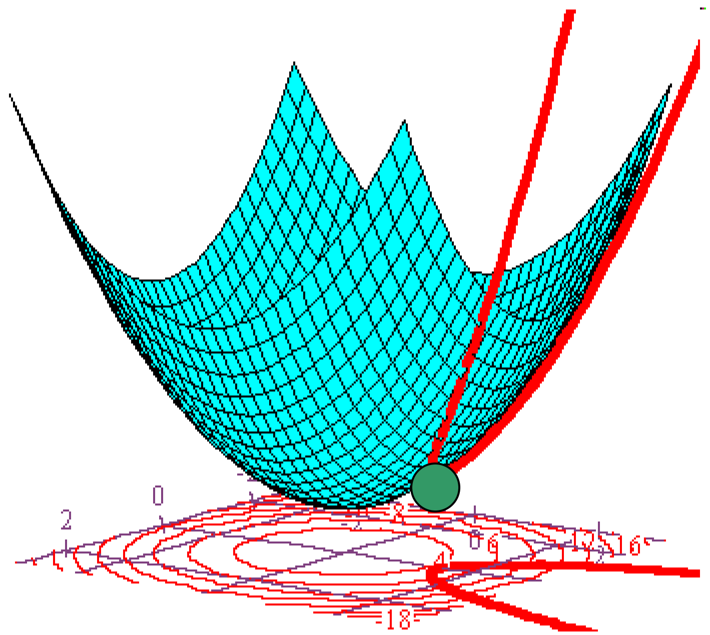
We can also think of this curve as representing a constraint on the values for *x* and *y* that we can plug into our function *z=f(x,y)*.



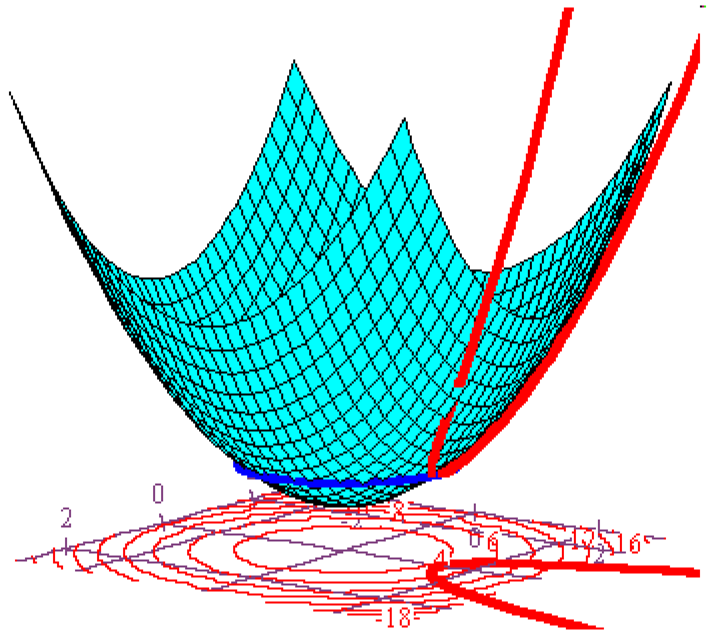
If we restrict the domain of *z=f(x,y)* to the curve *g(x,y)=c*, then the graph that results is just a curve lying on our original surface.



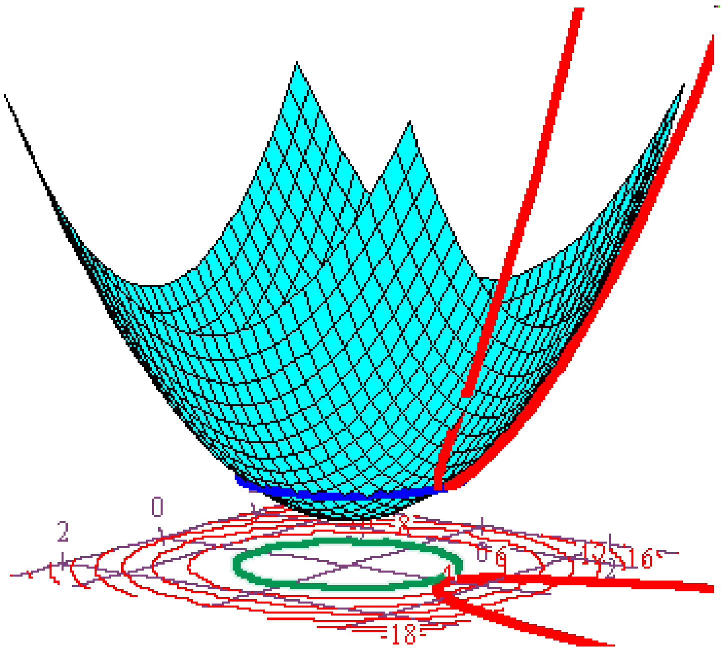
In this particular case, it’s easy to see that this curve has its own minimum point.

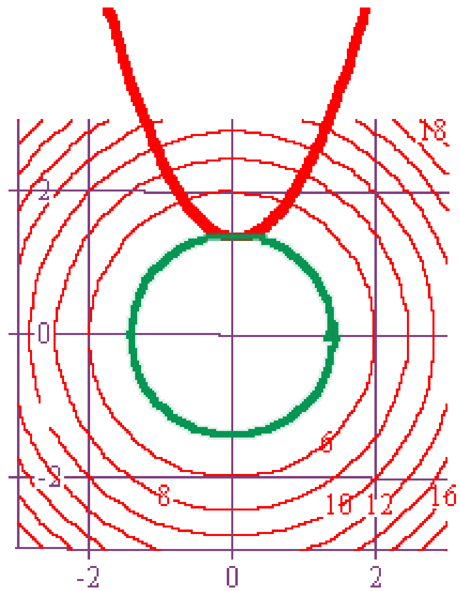


It's also easy to see that there is a contour, *z*=*k*, that touches our curve at that minimum point.

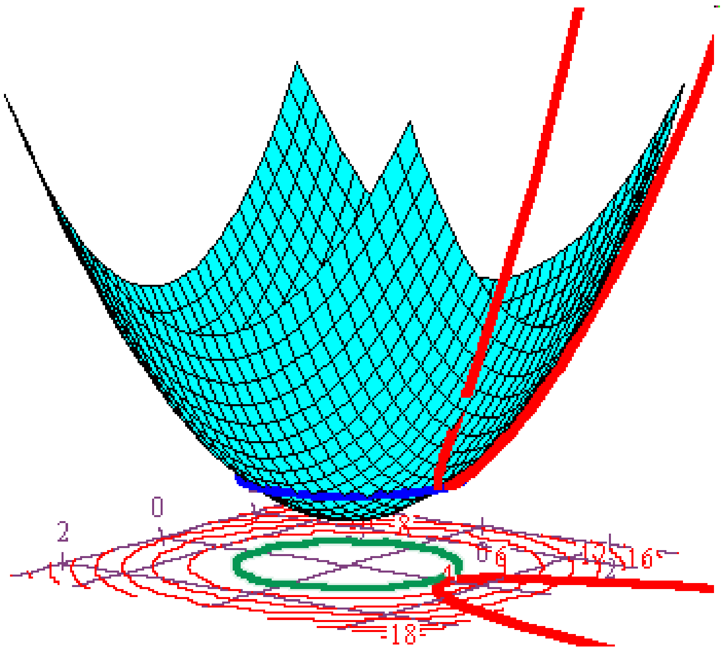


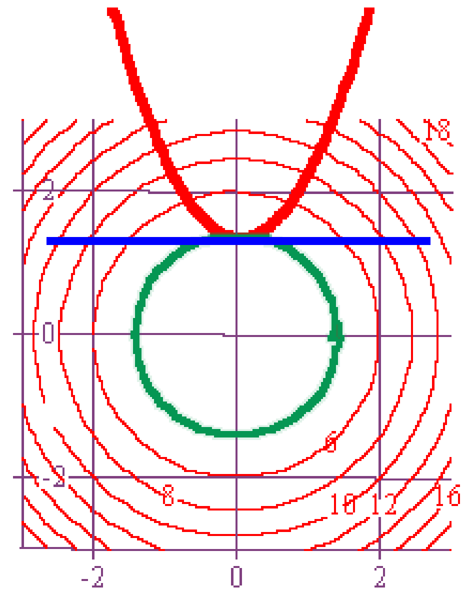
If we look at the level curve for this contour, we see that it is tangent to the curve *g(x,y)=c* in the *xy*-plane.



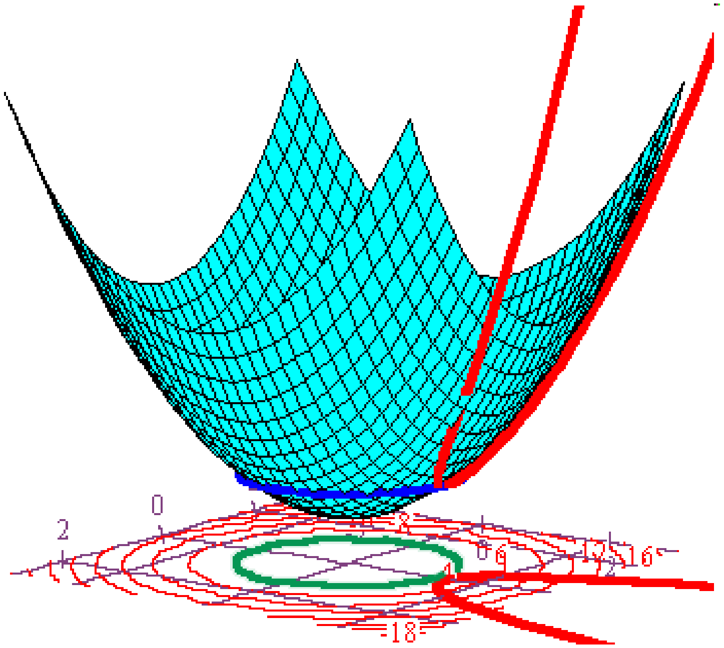


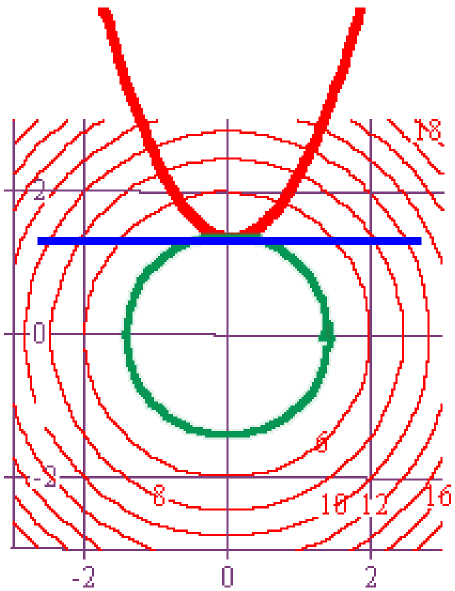
Hence, our level curve and *g(x,y)=c* have a common tangent line in the *xy*-plane.



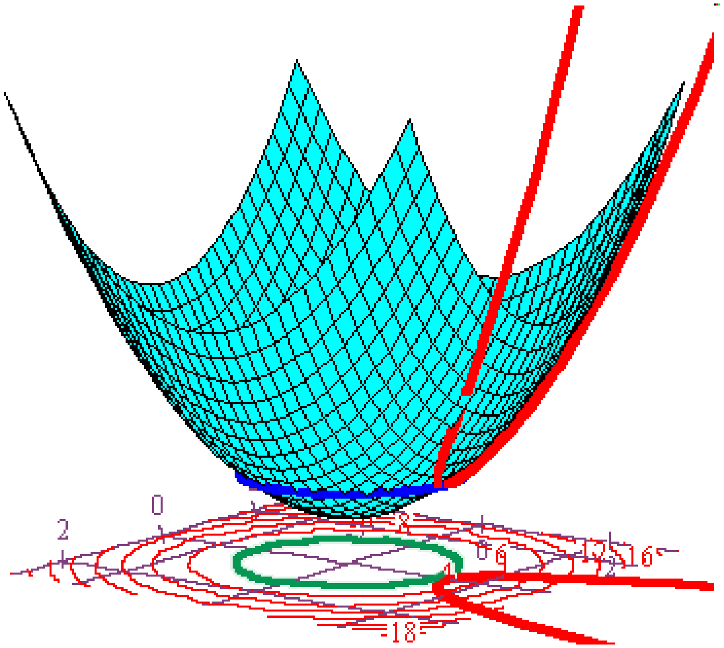


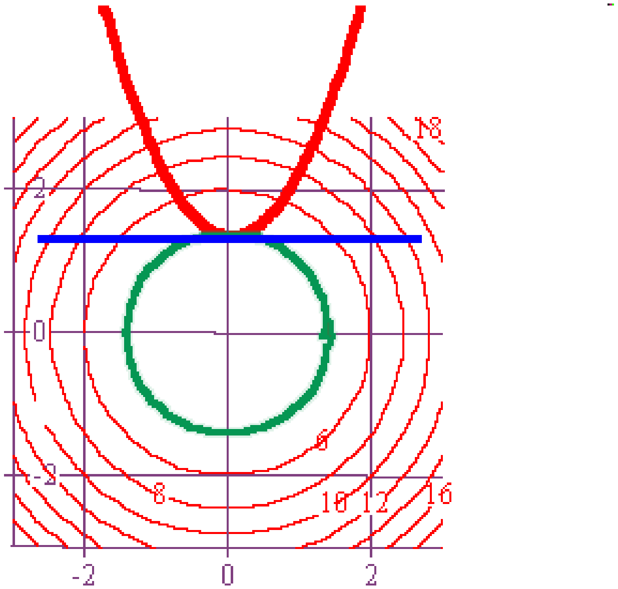
Let’s think about what this means.



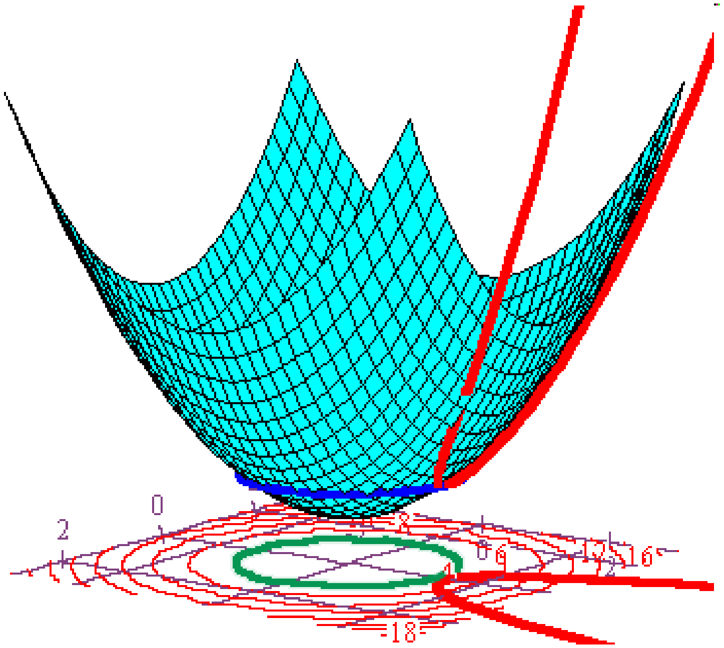


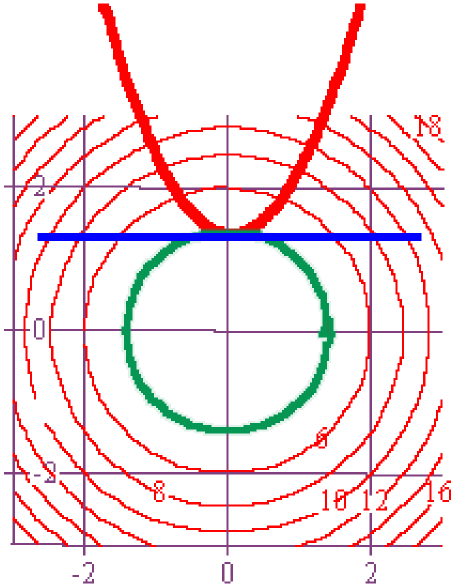
It must follow that both the gradient of *z* at this point and the gradient of *g* at this point are perpendicular to that tangent line.





Consequently, the gradient of *z* and the gradient of *g*, both evaluated at this point, are parallel.



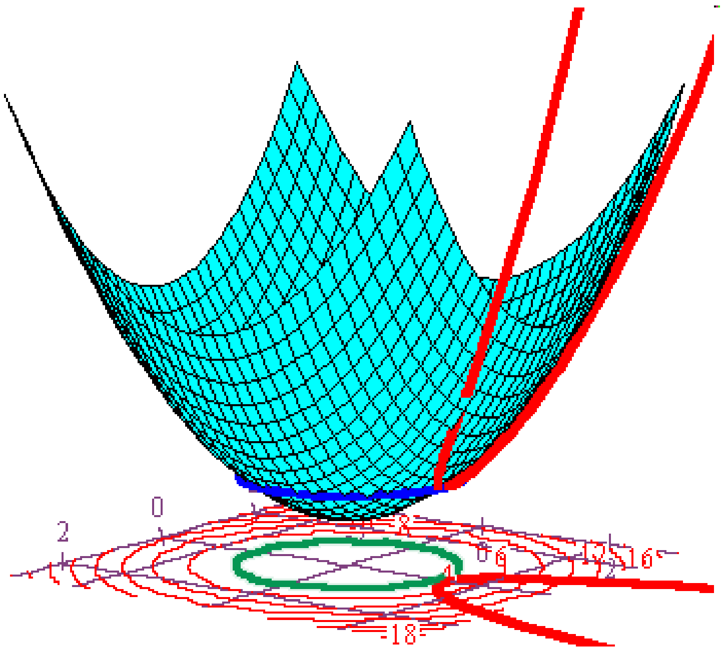


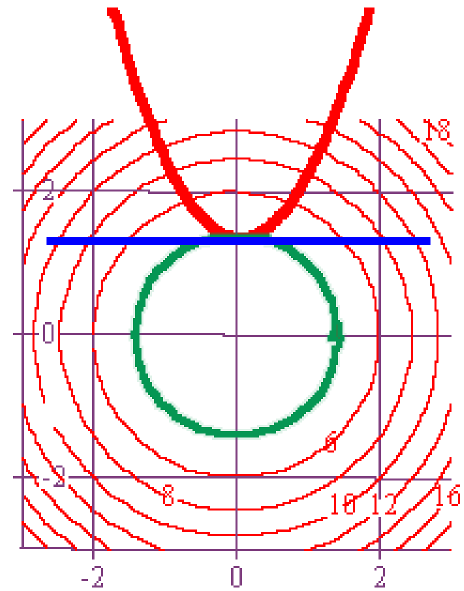
Therefore,

$$\nabla z = \lambda \nabla g$$

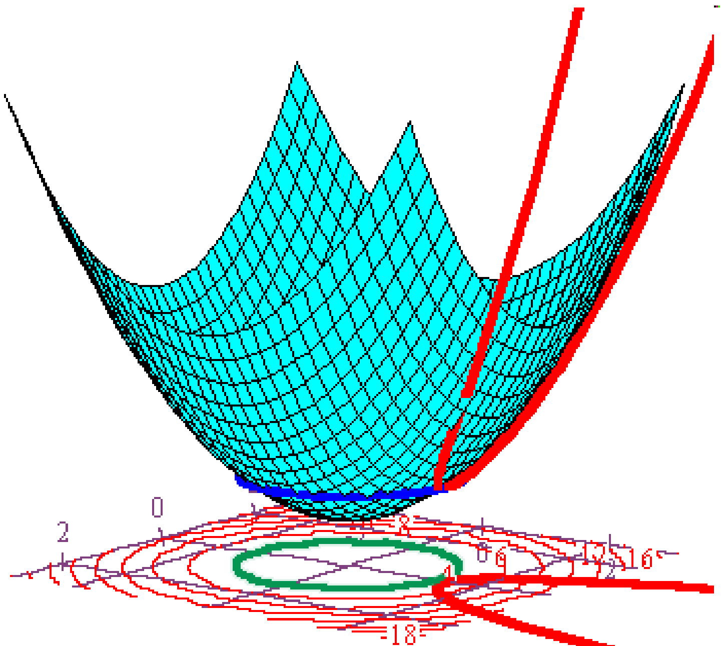
$$\Rightarrow \left(z_x \hat{i} + z_y \hat{j}\right) = \lambda\left(g_x \hat{i} + g_y \hat{j}\right)$$

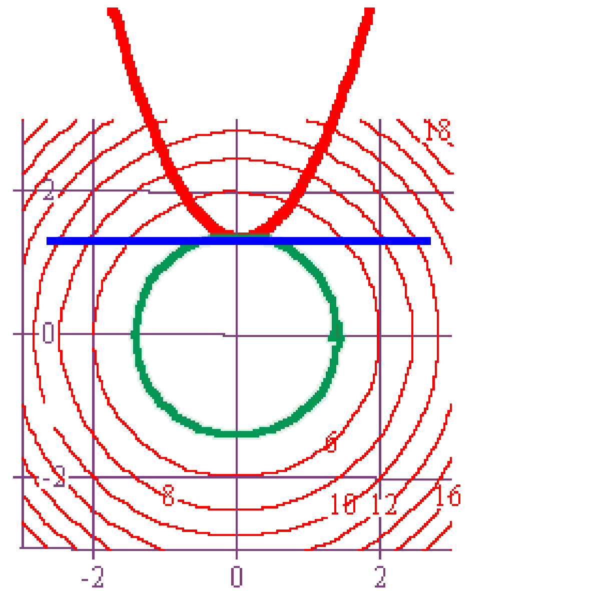
$$\Rightarrow z_x = \lambda g_x \ \ \& \ \ z_y = \lambda g_y$$





To find the the coordinates of the extreme point, you now just need to figure out how to solve the system of equations below.



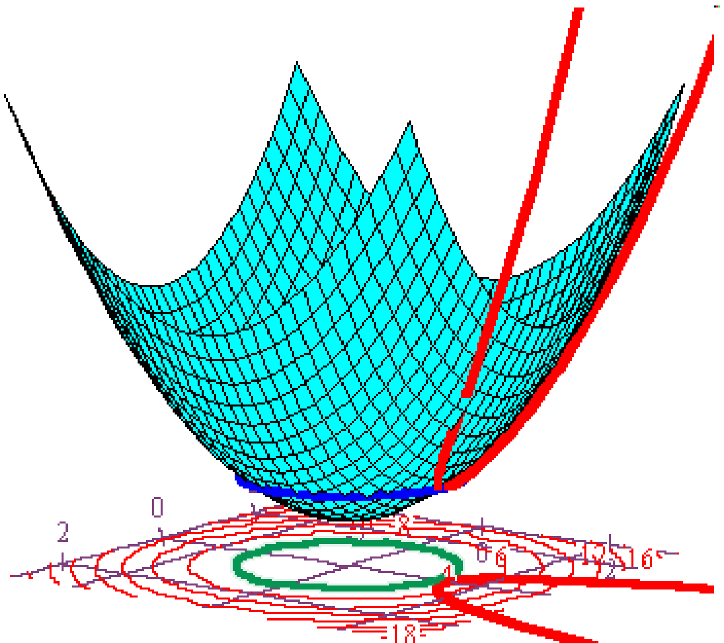


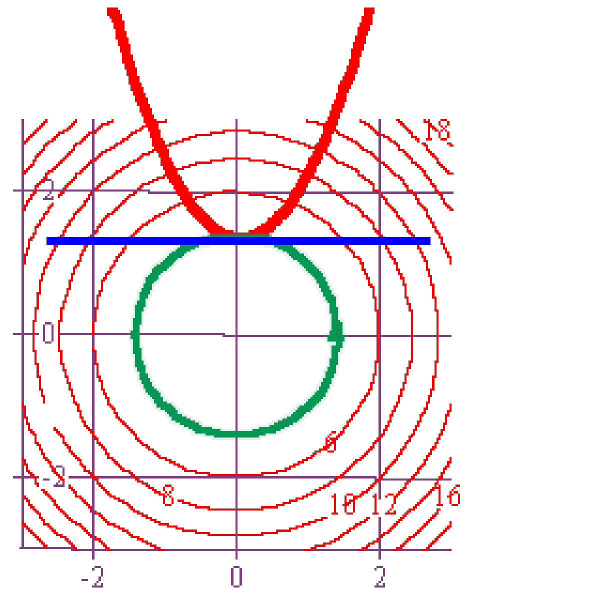
$$g(x,$$

$$z_x = \lambda g_x$$

$$z_y = \lambda g_y$$

GOOD LUCK!!!





$g(x$

$$z_x = \lambda g_x$$

$$z_y = \lambda g_y$$

Lagrange's Theorem: Let f and g have continuous first partial derivatives such that f has an extreme value at an interior point (x_0, y_0) on a smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \vec{0}$, then there is a real number λ such that $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$.

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PROOF: Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ be a smooth parametrization for the constraint curve, and suppose $f(x_0, y_0) = f(x(t_0), y(t_0))$ is an extreme value. Then since f is differentiable along this curve,

$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \bullet \frac{d\vec{r}}{dt}$ =0 when these derivatives are evaluated at $t = t_0$. Therefore, $\nabla \text{f}(x_0, y_0) \perp \vec{r}'(t_0)$. But since $\vec{r}(t)$ is a level curve for $\text{g} = g(x,y)$, $\nabla \text{g}\ (x_0, y_0)$ is also perpendicular to $\vec{r}'(t_0)$.

Therefore, $\nabla \text{f}(x_0, y_0) \,||\, \nabla g(x_0, y_0) \Rightarrow \nabla \text{f}(x_0, y_0) = \lambda \nabla g(x_0, y_0)$.