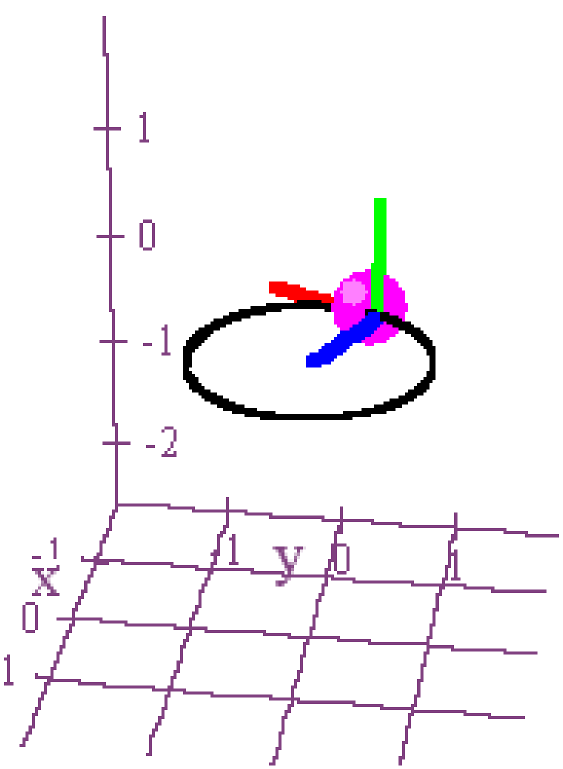
# MORE ON UNIT TANGENTS, UNIT NORMALS, AND BINORMALS



**If we are working in two dimensions, then the previous definition we gave for the unit normal vector is going to be the best one to use for the applications that we'll see later on.**

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$$N = N(t) = \frac{T'(t)}{\left\|T'(t)\right\|}$$

**Before going any further with this, however, we should prove that the vector *N* as defined below actually is perpendicular to *T*.**

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Proof: Recall that $\|T\| = 1$. Hence, $1 = \|T\|^2 = T \bullet T$. Thus,

$$0 = \frac{d(1)}{dt} = \frac{d\|T\|^2}{dt} = \frac{d(T \bullet T)}{dt} = T \bullet \frac{dT}{dt} + \frac{dT}{dt} \bullet T = 2\left(T \bullet \frac{dT}{dt}\right) \Rightarrow T \bullet \frac{dT}{dt} = 0$$

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From this proof it follows that the unit vector $N = \dfrac{T'}{\|T'\|}$ is also perpendicular to $T$.

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$$N = \frac{T'(t)}{\left\| T'(t) \right\|} = -\cos t\,\hat{i} - \sin t\,\hat{j}$$

**Notice that this definition for the unit normal gives us a result that is different from our original definition.**

$$T = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

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$$= -\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

**Remember, though, that in the applications that will come later on in the semester, we will always use our original definition.**

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$$T = \frac{\vec{r}\,'(t)}{\left\| r'(t) \right\|} = \frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\,\hat{j}$$

$$T'(t) = 0\,\hat{i} + 0\,\hat{j} = \vec{0}$$

**In this example, we can't even define a unit normal using the new definition.**

$$T = \frac{\vec{r}'(t)}{\left\| r'(t) \right\|} = \frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{j}$$

<u>Original</u>

$$N = \frac{2}{\sqrt{5}}\hat{i} - \frac{1}{\sqrt{5}}\hat{j}$$

<u>New</u>

$$N = \frac{T'}{\left\| T' \right\|} = \frac{\vec{0}}{0} = ?$$

**Again, for most of the applications we see, we will use the original definition for the unit normal in two dimensions.**

$$T = \frac{\vec{r}\,'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{j}$$

Original

$$N = \frac{2}{\sqrt{5}}\hat{i} - \frac{1}{\sqrt{5}}\hat{j}$$

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$$N = \frac{T'}{\|T'\|} = \frac{\vec{0}}{0} = ?$$

**Notice, however, that if we find *T* and *N* using our new definition, then the vector *B = T x N* is perpendicular to both our unit tangent and our unit normal vectors.**

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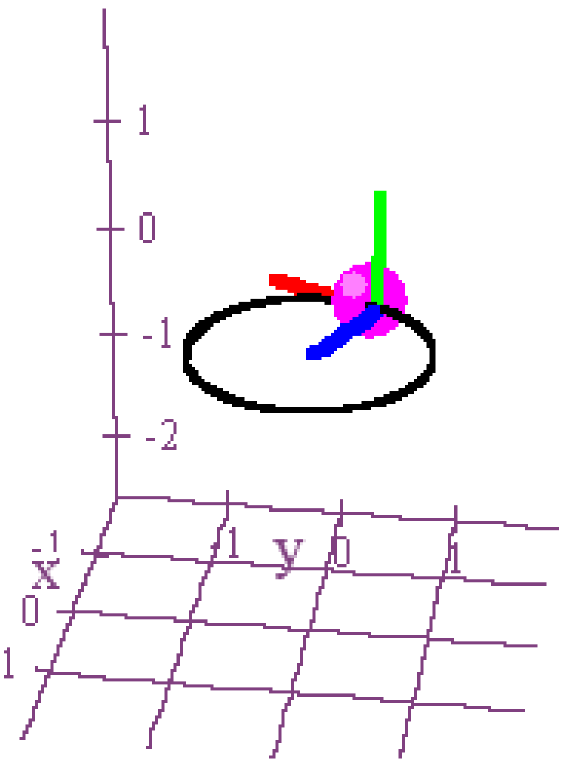
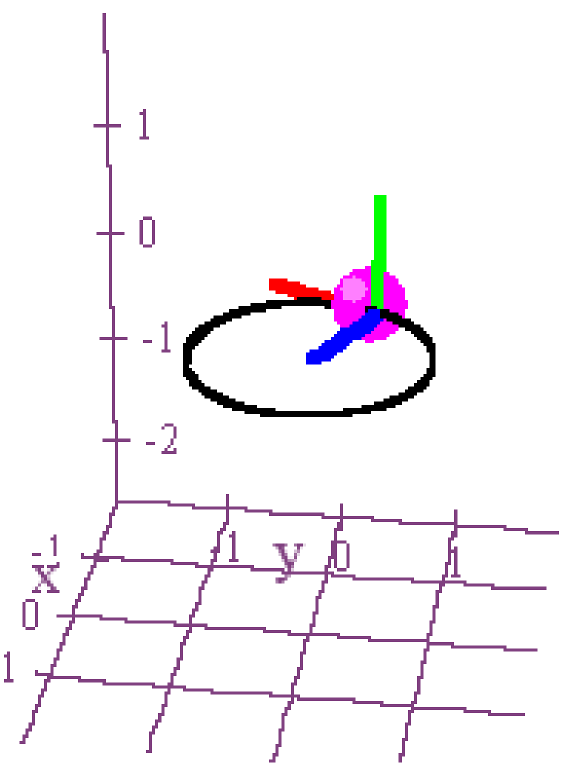
**We call *B* the *binormal vector.***

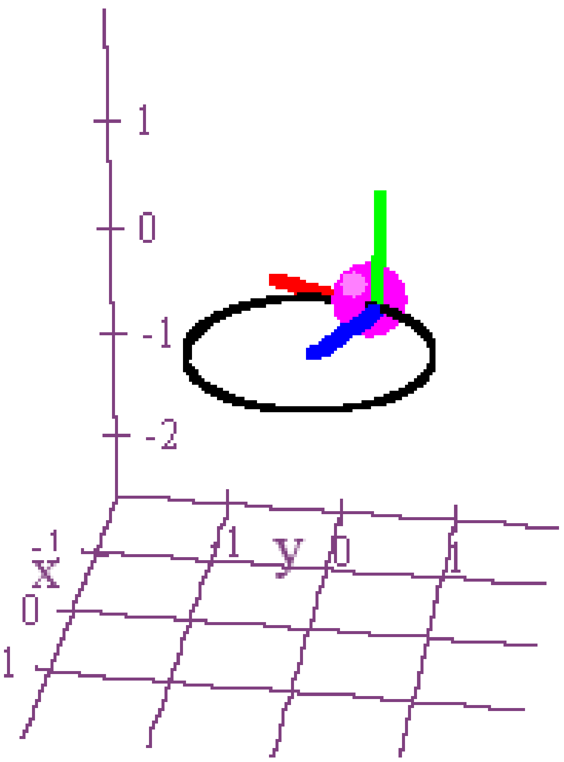


**The *binormal vector* is also a unit vector.**

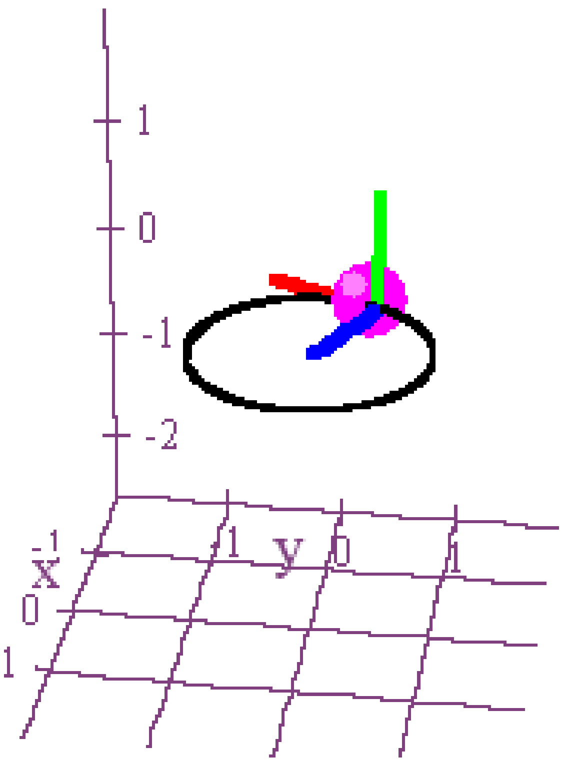
$$\|B\| = \|T \times N\| = \|T\|\|N\|\sin\frac{\pi}{2} = 1$$



**We also call the collective image of *T, N,* and *B* a *TNB* or *Frenet frame*.**



**The *TNB frame* does two things for us. First, it illustrates for us the preferred orientation for coordinate axes at a point.**



**And second, it shows us how the curve is twisting on itself as we go around it.**

$$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + \sin 3t\,\hat{k}$$