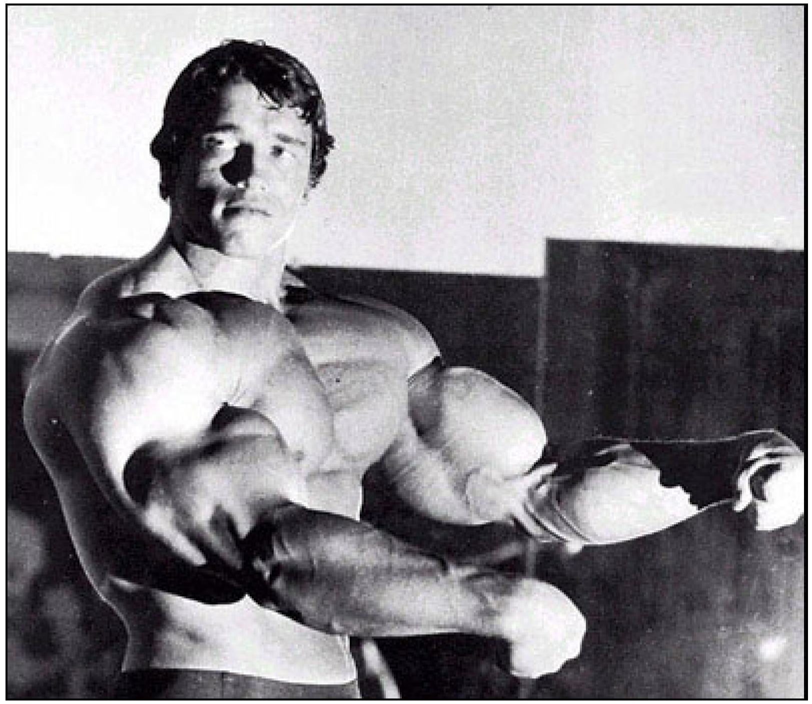
OPTIMIZATION



Definition: A function $z = f(x, y)$ has a global or absolute maximum on a region R at a point (a,b) if $f(a,b) \geq f(x, y)$ for all points (x, y) in R.

Definition: A function $z = f(x, y)$ has a global or absolute minimum on a region R at a point (a,b) if $f(a,b) \leq f(x, y)$ for all points (x, y) in R.

Definition: A point (a,b) is a boundary point of a region R if every disk centered at (a,b) contains both points in R and points not in R.

Definition: A point (a,b) is an interior point of a region R if it is not a boundary point of R.

Definition: The boundary of a region R is the set of all boundary points of R.

Definition: The interior of a region R is the set of all interior points of R.

Definition: A region R is closed if it contains all its boundary points.

Definition: A region R is open if every point is an interior point.

Definition: A region R is bounded if it can be contained inside some circle of sufficiently large radius k.

Theorem: If $z = f(x, y)$ is a continuous function defined on a closed and bounded region R, then $z = f(x, y)$ has both a global maximum and a global minimum value on the region R. These extreme values will occur either at critical points or at points on the boundary of R.

EXAMPLE 1: A company manufactures two items which are sold in two separate markets. The quantities q_1 and q_2 demanded by consumers and the prices p_1 and p_2, in dollars, of each item are related by,

$$p_1 = 600 - 0.3q_1$$
$$p_2 = 500 - 0.2q_2$$

The companies total production cost is,

$$C = 16 + 1.2q_1 + 1.5q_2 + 0.2q_1q_2$$

Find the maximum profit and how much of each product should be produced.

$$p_1 = 600 - 0.3q_1$$
$$p_2 = 500 - 0.2q_2$$

$$C = 16 + 1.2q_1 + 1.5q_2 + 0.2q_1q_2$$

$$\text{Revenue } = R = p_1q_1 + p_2q_2 = (600 - 0.3q_1)q_1 + (500 - 0.2q_2)q_2$$
$$= 600q_1 - 0.3q_1^2 + 500q_2 - 0.2q_2^2$$

$$\text{Profit } = P = R - C$$
$$= -0.3q_1^2 - 0.2q_2^2 - 0.2q_1q_2 + 598.8q_1 + 498.5q_2 - 16$$

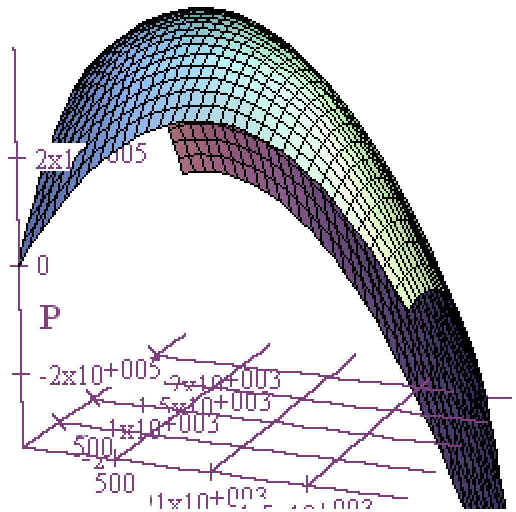
$$0 \le q_1 \le 2000$$
$$0 \le q_2 \le 2500$$

$$\text{Profit} = P = R - C$$

$$= -0.3q_1^2 - 0.2q_2^2 - 0.2q_1q_2 + 598.8q_1 + 498.5q_2 - 16$$

$$0 \le q_1 \le 2000$$

$$0 \le q_2 \le 2500$$



$$\frac{\partial P}{\partial q_1} = -0.6q_1 - 0.2q_2 + 598.8$$

$$\frac{\partial P}{\partial q_2} = -0.2q_1 - 0.4q_2 + 498.5$$

$$\begin{matrix} \frac{\partial P}{\partial q_1} = 0 \\ \frac{\partial P}{\partial q_2} = 0 \end{matrix} \Rightarrow \begin{matrix} 0.6q_1 + 0.2q_2 = 598.8 \\ 0.2q_1 + 0.4q_2 = 498.5 \end{matrix} \Rightarrow \begin{matrix} q_1 = 699.1 \\ q_2 = 896.7 \end{matrix}$$

$$\frac{\partial^2 P}{\partial {q_1}^2} = -0.6 \qquad \frac{\partial^2 P}{\partial q_2 \partial q_1} = -0.2$$

$$\frac{\partial^2 P}{\partial q_1 \partial q_2} = -0.2 \qquad \frac{\partial^2 P}{\partial {q_2}^2} = -0.4$$

Second Partials Test:

$$\frac{\partial^2 P}{\partial q_1^{\,2}} = -0.6 \qquad \frac{\partial^2 P}{\partial q_2 \partial q_1} = -0.2$$

$$\frac{\partial^2 P}{\partial q_1 \partial q_2} = -0.2 \qquad \frac{\partial^2 P}{\partial q_2^{\,2}} = -0.4$$

$$D = \begin{vmatrix} -0.6 & -0.2 \\ -0.2 & -0.4 \end{vmatrix} = (-0.6)(-0.4) - (-0.2)(-0.2) = 0.2 > 0$$

$$\frac{\partial^2 P(699.1, 896.7)}{\partial q_1^{\,2}} = -0.6 < 0 \Rightarrow \text{maximum}$$

$$(699.1, 896.7, \$432,797.02)$$

The Real Maximum:

$(699, 897, \$432, 797) \quad \Leftarrow$ Maximum

$(699, 896, \$432, 796.90)$

$(700, 896, \$432, 796.80)$

$(700, 897, \$432, 796.70)$

EXAMPLE 2: Twenty cubic meters of gravel are to be delivered to a landfill. The trucker plans to purchase an open-top box and make several trips. The box must have height 0.5m, but the trucker can choose the length and width. The cost of the box is $20 per square meter for the ends, and $10 per square meter for the sides and base. Each trip costs $2.00. Minimize the cost.

$x =$ side length

$y =$ end length

Volume = $0.5xy\,\text{m}^3$

Number of trips = $\dfrac{20}{0.5xy} = \dfrac{40}{xy}$

trip cost $= \dfrac{20}{0.5xy} \cdot 2 = \dfrac{80}{xy}$

side cost $= 2(0.5x) \cdot 10 = 10x$

end cost $= 2(0.5y) \cdot 20 = 20y$

bottom cost $= 10xy$

$x > 0$

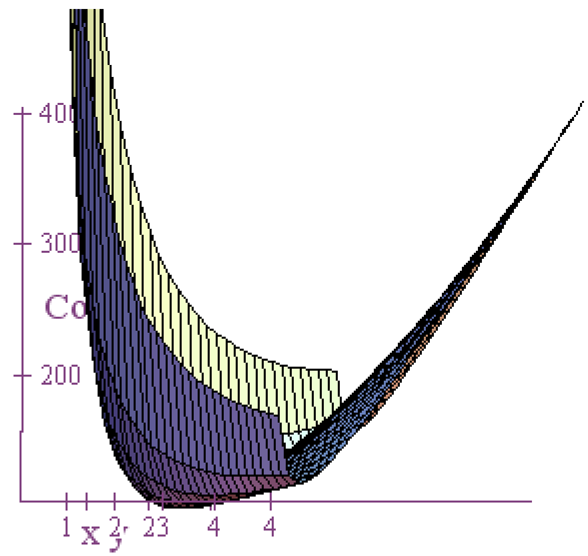
$y > 0$

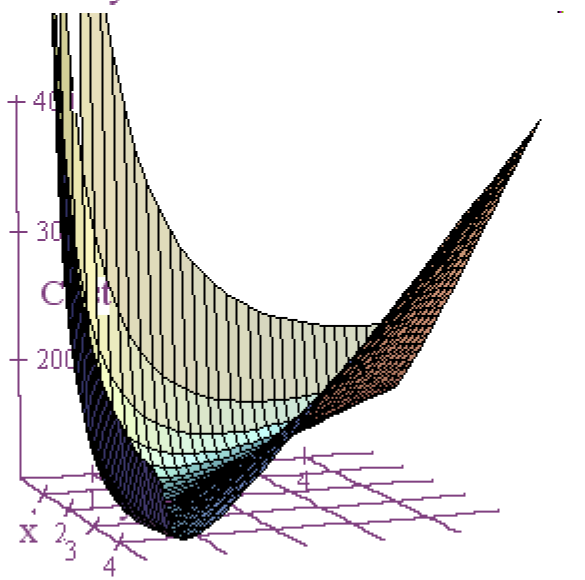
$$\text{Total Cost } = C = \frac{80}{xy} + 10x + 20y + 10xy$$

$$\text{Total Cost } = C = \frac{80}{xy} + 10x + 20y + 10xy$$

$$C_x = \frac{-80}{x^2 y} + 10 + 10y$$

$$C_y = \frac{-80}{xy^2} + 20 + 10x$$

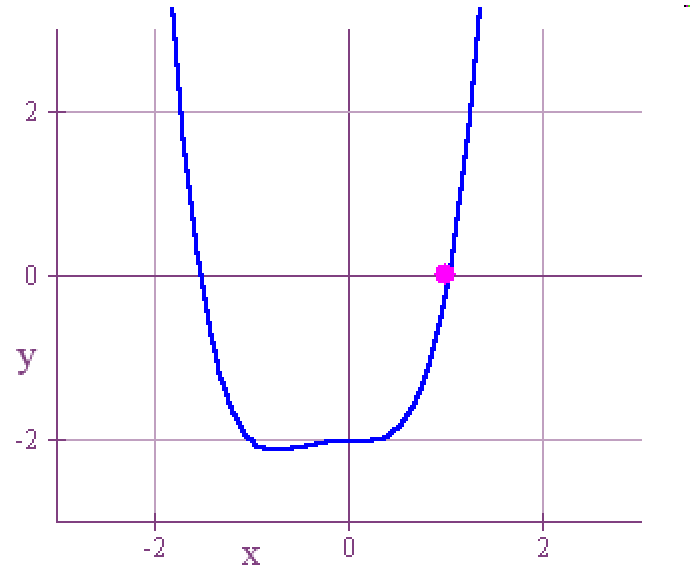




$$\begin{aligned} C_x &= 0 \\ C_y &= 0 \end{aligned} \Rightarrow \begin{aligned} &\frac{-80}{x^2 y} + 10 + 10y = 0 \\ &\frac{-80}{xy^2} + 20 + 10x = 0 \end{aligned} \Rightarrow \begin{aligned} 1 + y &= \frac{8}{x^2 y} \\ 2 + x &= \frac{8}{xy^2} \end{aligned}$$

$$\frac{1+y}{2+x} = \frac{\frac{8}{x^2 y}}{\frac{8}{xy^2}} = \frac{y}{x} \Rightarrow x + xy = 2y + xy \Rightarrow x = 2y$$

$$\Rightarrow 1 + y = \frac{8}{4y^3} = \frac{2}{y^3} \Rightarrow y^4 + y^3 - 2 = 0$$



By inspection, $y^4 + y^3 - 2 = 0$ when $y = 1$.

Hence, the critical point is $x = 2$ and $y = 1$.

$$C_x = \frac{-80}{x^2 y} + 10 + 10y \qquad C_{xx} = \frac{160}{x^3 y} \qquad C_{xy} = \frac{80}{x^2 y^2} + 10$$

$$C_y = \frac{-80}{xy^2} + 20 + 10x \qquad C_{yx} = \frac{80}{x^2 y^2} + 10 \qquad C_{yy} = \frac{160}{xy^3}$$

$$D = \frac{160^2}{x^4 y^4} - \left(\frac{80}{x^2 y^2} + 10 \right)^2 = \frac{160^2}{x^4 y^4} - \frac{80^2}{x^4 y^4} - \frac{1600}{x^2 y^2} - 100$$

$$= \frac{19,200}{x^4 y^4} - \frac{1600}{x^2 y^2} - 100$$

$$D(2,1) = \frac{19,200}{2^4} - \frac{1600}{2^2} - 100 = 700$$

$$C_{xx}(2,1) = \frac{160}{2^3} = 20 > 0 \Rightarrow \text{minimum}$$

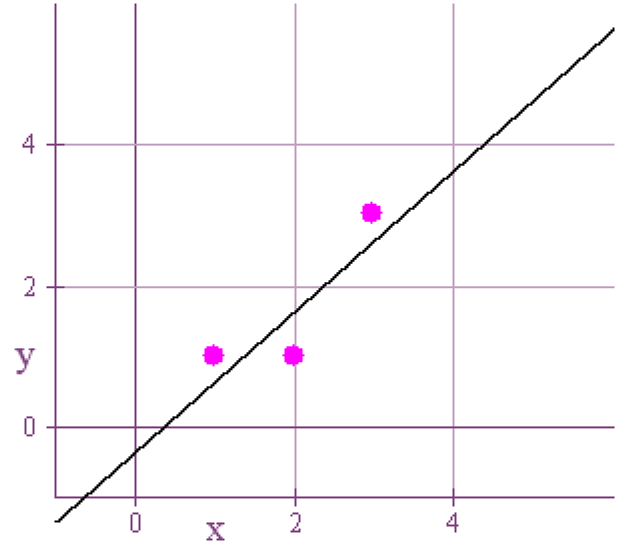
$$\text{Total Cost } = C = \frac{80}{xy} + 10x + 20y + 10xy$$

Let the length $x = 2$ meters and the width $y = 1$ meter.

This results in a minimum cost of

$$C(2,1) = \frac{80}{2} + 10 \cdot 2 + +20 \cdot 1 + 10 \cdot 2 = \$100.00.$$

EXAMPLE 3: Find the least squares regression line that best fits the points $(1,1), (2,1),$ and $(3,3)$.



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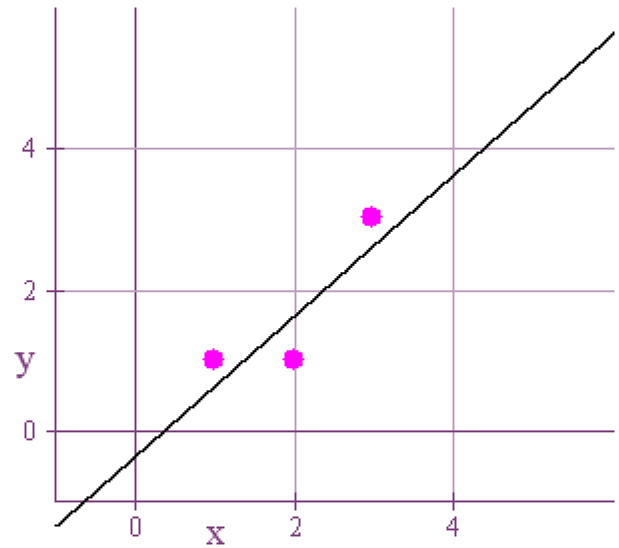

The line has equation $y = mx + b$.

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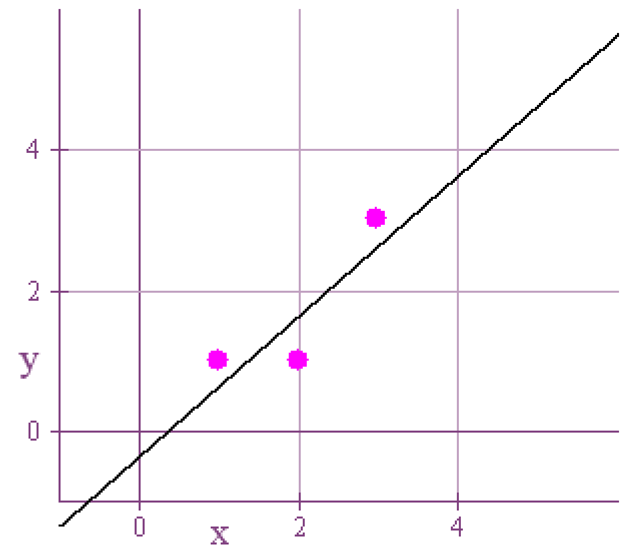

Corresponding points on the line are:

$(1, m+b)$

$(2, 2m+b)$

$(3, 3m+b)$

We want to minimize the squares of the vertical distances from the points to the line.

$$f(m,b) = (m+b-1)^2 + (2m+b-1)^2 + (3m+b-3)^2$$

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$$f(m,b) = (m+b-1)^2 + (2m+b-1)^2 + (3m+b-3)^2$$

$$f_m = 2(m+b-1) + 2(2m+b-1)\cdot 2 + 2(3m+b-3)\cdot 3$$
$$= 28m + 12b - 24$$

$$f_b = 2(m+b-1) + 2(2m+b-1) + 2(3m+b-3)$$
$$= 12m + 6b - 10$$

$$\begin{matrix} f_m = 0 \\ f_b = 0 \end{matrix} \Rightarrow \begin{matrix} 28m + 12b - 24 = 0 \\ 12m + 6b - 10 = 0 \end{matrix} \Rightarrow \begin{matrix} m = 1 \\ b = -\dfrac{1}{3} \end{matrix}$$

$$f_m = 28m + 12b - 24$$

$$f_b = 12m + 6b - 10$$

$$\text{critical point} = (1, -1/3)$$

$$f_{mm} = 28 \qquad f_{mb} = 12$$
$$f_{bm} = 12 \qquad f_{bb} = 6$$

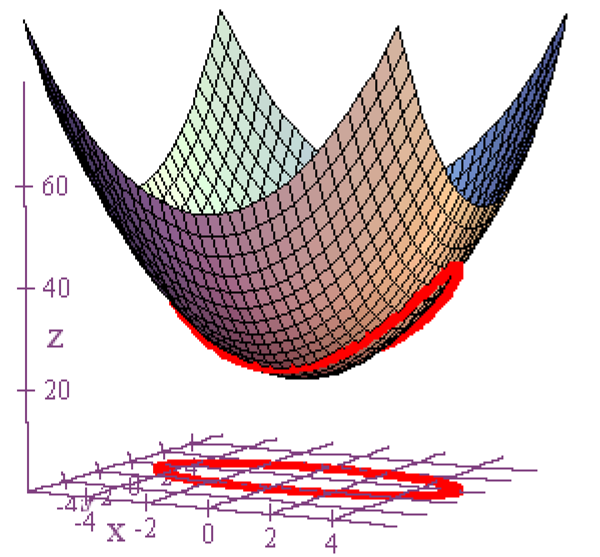
$$D\left(1, -\frac{1}{3}\right) = \begin{vmatrix} 28 & 12 \\ 12 & 6 \end{vmatrix} = 28 \cdot 6 - 12 \cdot 12 = 24 > 0$$

$$f_{mm}\left(1, -\frac{1}{3}\right) = 28 > 0 \Rightarrow \text{ minimum}$$

$$\text{regression line: } y = x - \frac{1}{3}$$

EXAMPLE 4: Find the maximum and minimum values of $z = f(x, y) = x^2 + y^2 + 20$ on the region R that has the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$ as its boundary.

Because our function is continuous on a closed and bounded region, we are guaranteed that both a global maximum and minimum will exist.



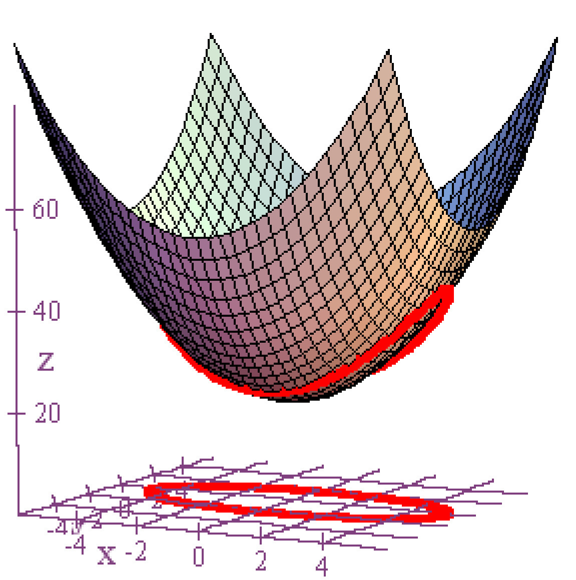
Furthermore, examination of the graph suggests that the minimum will occur at an interior point, and the maximum will occur at the points $(-5, 0)$ and $(5, 0)$.

For the minimum point, use the second partials test.

$$z = x^2 + y^2 + 20$$

$$z_x = 2x \qquad z_{xx} = 2 \quad z_{xy} = 0$$

$$z_y = 2y \qquad z_{yx} = 0 \quad z_{yy} = 2$$



$$\left.\begin{array}{l} z_x = 0 \\ z_y = 0 \end{array}\right. \Rightarrow \begin{array}{l} 2x = 0 \\ 2y = 0 \end{array} \Rightarrow \begin{array}{l} x = 0 \\ y = 0 \end{array}$$

$$D(0,0) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2 \cdot 2 - 0 \cdot 0 = 4 > 0$$

$z_{xx}(0,0) = 2 > 0 \Rightarrow$ minimum

$(0,0,20)$ is a minimum point.

Maximum points are $(-5,0,45)$ and $(5,0,45)$.