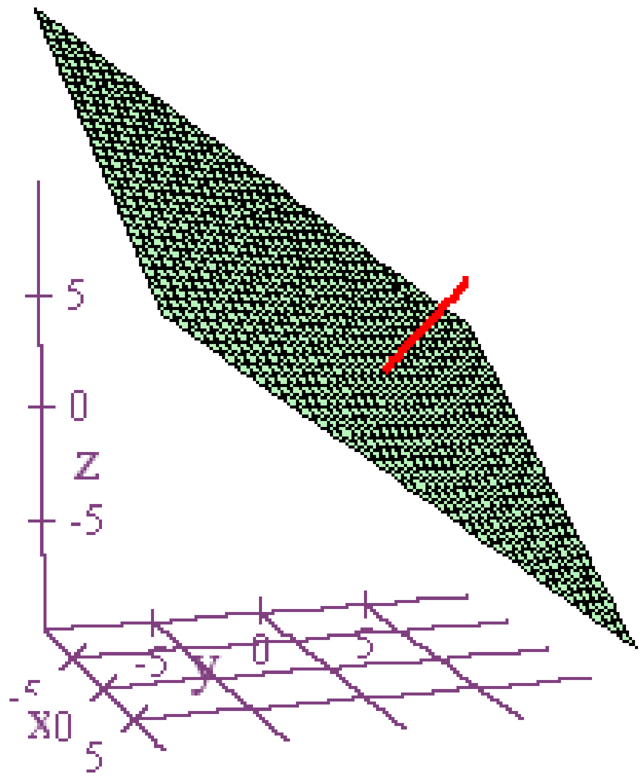
THE PLANE TRUTH



Recall that $z = m_x x + m_y y + c$ is an equation for a plane.

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In general, an equation for a plane can be written as

$Ax + By + Cz + D = 0$

where not all of the coefficients $A, B,$ and C equal zero.

Suppose that we are given $2x+3y+4z-20=0$.

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Then we claim that the vector $\vec{v} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ is perpendicular to this plane.

To see this, notice that $(1, 2, 3)$ is a point in this plane.

$$2x + 3y + 4z - 20 = 0$$

$$\vec{v} = 2\hat{i} + 3\hat{j} + 4\hat{k} \qquad P = (1, 2, 3)$$

If $Q = (x, y, z)$ is any other point in this plane, then the displacement vector from P to Q is:

$$\overrightarrow{PQ} = (x - 1)\hat{i} + (y - 2)\hat{j} + (z - 3)\hat{k}$$

$$2x + 3y + 4z - 20 = 0$$

$$\vec{v} = 2\hat{i} + 3\hat{j} + 4\hat{k} \qquad P = (1, 2, 3)$$

$$\overrightarrow{PQ} = (x-1)\hat{i} + (y-2)\hat{j} + (z-3)\hat{k}$$

Furthermore, $\vec{v} \bullet \overrightarrow{PQ} = 2(x-1) + 3(y-2) + 4(z-3)$

$= 2x - 2 + 3y - 6 + 4z - 12$

$= 2x + 3y + 4z - 20 = 0$

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$$\vec{v} = 2\hat{i} + 3\hat{j} + 4\hat{k} \qquad P = (1, 2, 3)$$

$$\overrightarrow{PQ} = (x-1)\hat{i} + (y-2)\hat{j} + (z-3)\hat{k}$$

Therefore, $\vec{v} \perp \overrightarrow{PQ}$, i.e. $\vec{v}$ is perpendicular to $\overrightarrow{PQ}$.

$$2x + 3y + 4z - 20 = 0$$

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Since the point Q in the plane was picked arbitrarily,

$\vec{v}$ is perpendicular to any vector $\overrightarrow{PQ}$ in the plane

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Therefore, $\vec{v}$ is perpendicular

to the plane $2x + 3y + 4y - 20 = 0.$

Similary, if $\vec{v} = A\hat{i} + B\hat{j} + C\hat{k}$ is a vector, and if $P = (a, b, c)$ is a point, then the plane containing P that is perpendicular to $\vec{v}$ consists of all points $Q = (x, y, z)$ such that $\vec{v} \bullet \overrightarrow{PQ} = 0.$

But $\vec{v} \bullet \overrightarrow{PQ} = \left(A\hat{i} + B\hat{j} + C\hat{k} \right) \bullet \left((x-a)\hat{i} + (y-b)\hat{j} + (z-c)\hat{k} \right)$

$= A(x-a) + B(y-b) + C(z-c)$

$= Ax + By + Cz + A(-a) + B(-b) + C(-c) =$

$= Ax + By + Cz + D = 0.$

In other words, whenever we have a plane $Ax + By + Cz + D = 0$, the vector $\vec{v} = A\hat{i} + B\hat{j} + C\hat{k}$ will be perpendicular to this plane.

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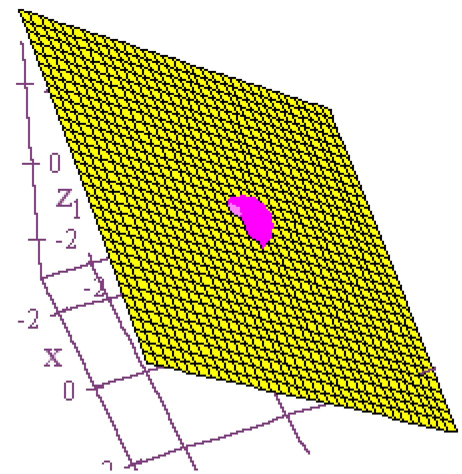
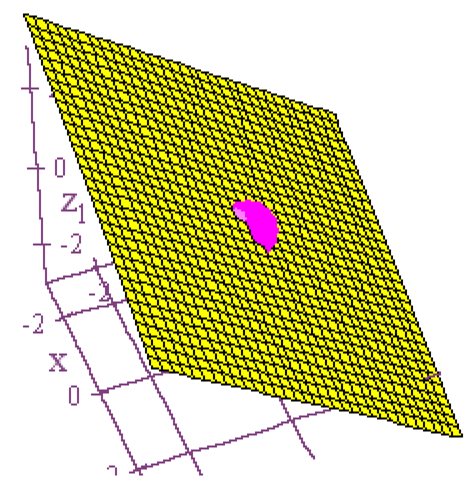
any plane $Ax + By + Cz + D = 0$ will be perpendicular to it.

Another way to look at this is to start with a plane that contains the point $(0,0,0)$, such as 2x+3y+4z=0.



Then if (x, y, z) is a point in this plane, it follows that

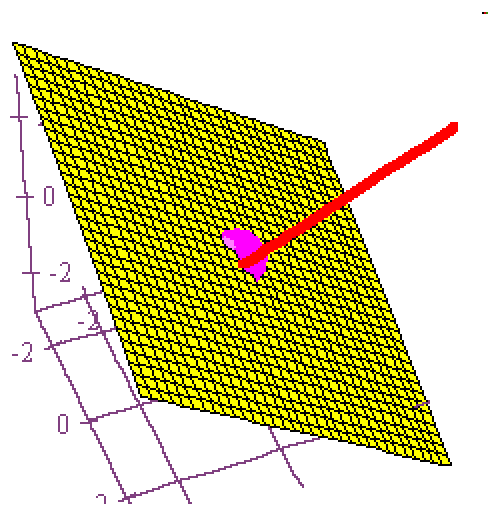
$\vec{w} = x\hat{i} + y\hat{j} + z\hat{k}$ is a vector that lies in this plane.



We can now write

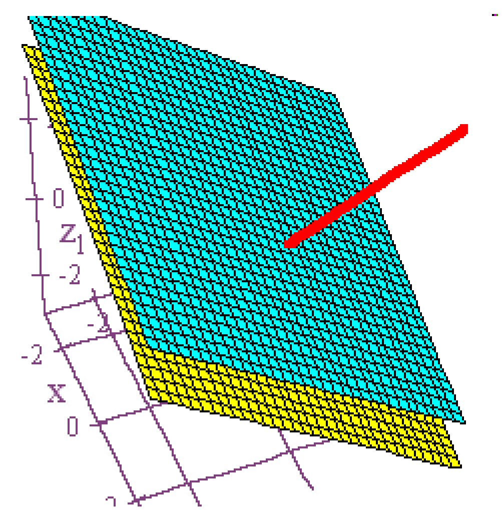
$$2x+3y+4z=\left(2\hat{i}+3\hat{j}+4\hat{k}\right)\bullet\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=0$$

$$\Rightarrow\left(2\hat{i}+3\hat{j}+4\hat{k}\right)\perp\left(x\hat{i}+y\hat{j}+z\hat{k}\right).$$

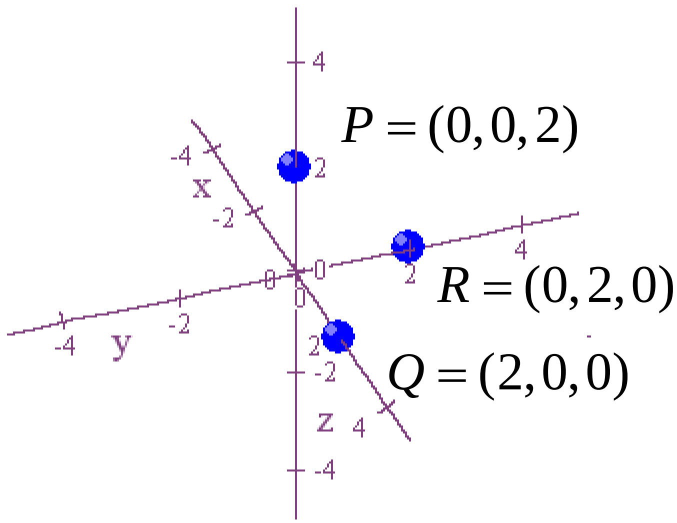


And finally, if we change the equation to $2x+3y+4z=5$, then all we have changed is the z-intercept, and

$2\hat{i}+3\hat{j}+4\hat{k}$ is still perpendicular to the plane.

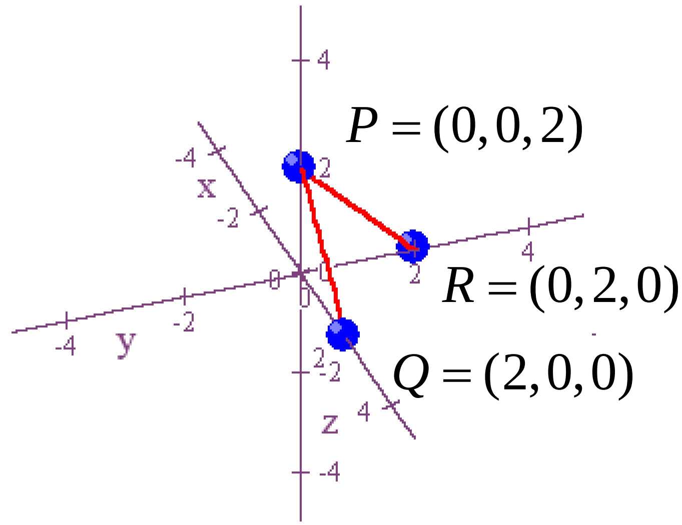


Now let's start with three noncollinear points in space.



Add the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

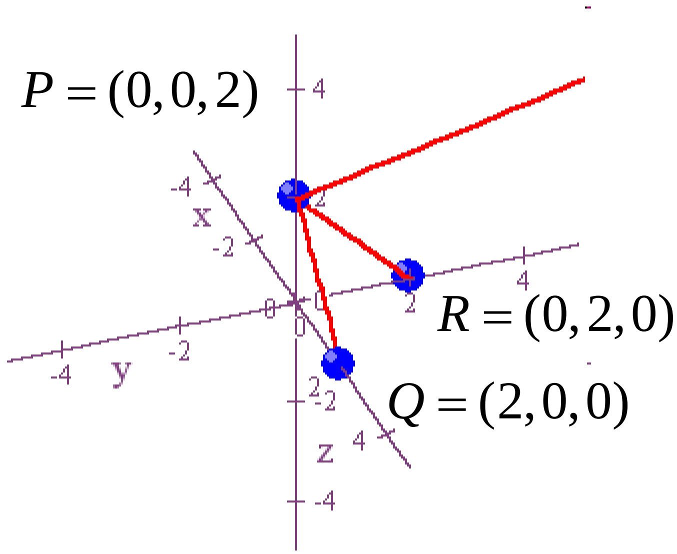
$\overrightarrow{PQ} = \langle 2, 0, -2 \rangle$



$\overrightarrow{PR} = \langle 0, 2, -2 \rangle$

Find the cross product $\overrightarrow{N} = \overrightarrow{PQ} \times \overrightarrow{PR}$.

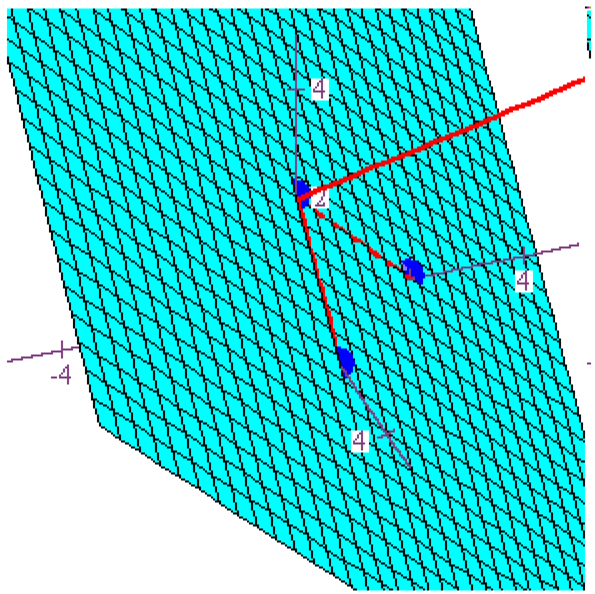
$\overrightarrow{PQ} = \langle 2, 0, -2 \rangle$



$\overrightarrow{PR} = \langle 0, 2, -2 \rangle$

$$\overrightarrow{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -2 \\ 0 & 2 & -2 \end{vmatrix} = 4\hat{i} + 4\hat{j} + 4\hat{k} = \langle 4, 4, 4 \rangle$$

Let $\overrightarrow{PS} = \langle x, y, z-2 \rangle$ be the vector from $P = (0,0,2)$ to an arbitrary point $S = (x, y, z)$ in the plane, and find an equation for the plane by computing $\overrightarrow{PS} \bullet \overrightarrow{N}$.



$\overrightarrow{N} = \langle 4,4,4 \rangle$

$\overrightarrow{PQ} = \langle 2,0,-2 \rangle$

$\overrightarrow{PR} = \langle 0,2,-2 \rangle$

$$\overrightarrow{PS} \bullet \overrightarrow{N} = 4x + 4y + 4z - 8 = 0 \Rightarrow z = -x - y + 2$$