

SECRET IDENTITIES



Consider the curl of a vector field F .

$$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$$

$$\begin{aligned} \text{curl of } \vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix} \hat{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix} \hat{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \hat{k} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} \end{aligned}$$

Now take the divergence of the curl.

$$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$$

$$\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

$$\text{div}(\text{curl}F) = \nabla \cdot (\nabla \times \vec{F}) = \nabla \cdot \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} \right]$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$$

Now take the divergence of the curl.

$$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$$

$$\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

$$\text{div}(\text{curl } \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = \nabla \cdot \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} \right]$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$$

$$\text{div}(\text{curl } \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$$

Now let $f = f(x, y, z)$.

$$f = f(x, y, z)$$

$$\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

Now take the curl of the gradient.

$$f = f(x, y, z)$$

$$\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

$$\text{curl of } \nabla f = \nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \hat{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \end{vmatrix} \hat{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix} \hat{k}$$

$$= (f_{zy} - f_{yz}) \hat{i} - (f_{xz} - f_{zx}) \hat{j} + (f_{xy} - f_{yx}) \hat{k} = \vec{0}$$

Now take the curl of the gradient.

$$f = f(x, y, z)$$

$$\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

$$\text{curl of } \nabla f = \nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \hat{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \end{vmatrix} \hat{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix} \hat{k}$$

$$= (f_{zy} - f_{yz}) \hat{i} - (f_{xz} - f_{zx}) \hat{j} + (f_{xy} - f_{yx}) \hat{k} = \vec{0}$$

$$\text{curl}(\nabla f) = \nabla \times (\nabla f) = \vec{0}$$